Does Preference Always Help? A Holistic Study on Preference-Based Evolutionary Multi-Objective Optimisation Using Reference Points*

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1 Introduction

It is not uncommon that real-world decision problems require solutions to simultaneously meet multiple objectives, known as multi-objective optimisation problems (MOPs). Note that these objectives are conflicting where an improvement in one objective can lead to a detriment of other objective(s). Hence, there does not exist a global optimum that optimises all objectives simultaneously. Instead, there exists a set of solutions representing the trade-offs among conflicting objectives. Generally speaking, a minimisation MOP considered in this paper is defined as follows:

$$\begin{array}{ll} \text{minimise} & \mathbf{F}(\mathbf{x}) = (f_1(\mathbf{x}), \cdots, f_m(\mathbf{x}))^T \\ \text{subject to} & \mathbf{x} \in \Omega \end{array}, \tag{1}$$

where $\mathbf{x} = (x_1, \ldots, x_n)^T$ is a decision vector and $\mathbf{F}(\mathbf{x})$ is a objective vector. $\Omega = [x_i^L, x_i^U]^n \subseteq \mathbb{R}^n$ defines the search space. $\mathbf{F} : \Omega \to \mathbb{R}^m$ is the corresponding attainable set in the objective space $\mathbb{R}^m$. Without considering any preference information from a decision maker (DM), given two solutions $\mathbf{x}^1, \mathbf{x}^2 \in \Omega$, $\mathbf{x}^1$ is said to dominate $\mathbf{x}^2$ if and only if $f_i(\mathbf{x}^1) \leq f_i(\mathbf{x}^2)$ for all $i \in \{1, \cdots, m\}$ and $\mathbf{F}(\mathbf{x}^1) \neq \mathbf{F}(\mathbf{x}^2)$. A solution $\mathbf{x} \in \Omega$ is said to be Pareto-optimal if and only if there is no solution $\mathbf{x}' \in \Omega$ that dominates it. The set of all Pareto-optimal solutions is called the Pareto set (PS) and their corresponding objective vectors form the Pareto front (PF).

Due to the population-based property, evolutionary algorithms (EAs) have been widely recognised as a major approach for MO. Over the past three decades and beyond, many efforts have been dedicated to developing evolutionary multi-objective optimisation (EMO) algorithms, such as non-dominated sorting genetic algorithm II (NSGA-II) [1], indicator-based EA (IBEA) [2] and multi-objective EA based on decomposition (MOEA/D) [3], to find a set of well-converged and well-diversified efficient solutions that approximate the whole PF. Nevertheless, the ultimate goal of MO is to help the DM identify a handful of representative solutions that meet at most her/his preferences. This inspires the requirement to incorporate the DM's preference information into MO – techniques have been studied in the multi-criterion decision-making (MCDM) community over half a century. There are

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three classes of hybrid techniques considering the synergy of EMO and MCDM: *a posteriori*, *a priori*, and *interactive*.

The traditional EMO follow the *a posteriori* decision-making where a set of widely spread trade-off alternatives are obtained by an EMO algorithm before being presented to the DM. However, this not only increases the DM's workload, but also provides much irrelevant or even noisy information during the decision-making process. Due to the curse of dimensionality, the performance of EMO algorithms degenerate with the number of objectives [4]. In addition, the number of points used to represent a PF grows exponentially with the number of objectives, thereby increasing the computational burden of an EMO algorithm. Besides, there is a severe cognitive obstacle for the DM to comprehend a high-dimensional PF.

If the preference information is elicited *a priori*, it is used as a criterion to evaluate the fitness of a solution in the environmental selection and to drive the population towards the region(s) of interest (ROI) along a pre-defined 'preferred' direction. In particular, the preference information can be represented as one or more reference points [5–9], reference directions [10], light beams [11] or value functions (VFs) [12]. Note that, in the *a priori* approach, the DM only interact with the algorithm at the outset of an EMO process. It is controversial that the DM is able to faithfully represent her/his preference information before solving the MOP at hand.

As for the *interactive* preference elicitation, it enables the DM to progressively learn and understand the characteristics of the MOP at hand and adjust her/his elicited preference information. Consequently, solutions are gradually driven towards the ROI. In principle, many *a priori* EMO approaches can be used in an interactive manner (e.g., [10] and [11]). Specifically, in the first round, the DM can elicit certain preference information and it is used in an EMO algorithm to find a set of preferred non-dominated solutions. Thereafter, a few representative solutions will be presented to the DM. If these solutions are satisfactory, they will be used as the outputs and the iterative procedure terminates. Otherwise, the DM will adjust her/his preference information accordingly and it will be used in another EMO run. Alternatively, the DM can be involved to periodically provide her/his preference information as the EMO iterations are underway [13]. In particular, the preference information is progressively learned as VFs with the evolution of solutions. Since the DM gets a more frequent chance to provide new information, as discussed in [14], the DM may feel more in charge and more involved in the overall *optimization-cum-decision-making* process.

Although many efforts have been devoted to the synergy of EMO and MCDM, there is no systematic study, at least to the best of our knowledge, to investigate the *pros and cons* brought by preference incorporation in EMO for approximating the ROI. This might be because although all preference-based EMO algorithms claim to approximate a ROI, the definition of the ROI is vague. In principle, it depends on the way how the DM elicits her/his preference information. For example, if the DM's preference information is elicited as a reference point, the ROI corresponds to a PF segment 'close' to this reference point. On the other hand, if the DM's preference information is elicited as a VF learned from pair-wise comparisons made by the DM, it is difficult to define a specific location of the ROI on the PF. Instead, the preferred solutions are subjectively determined by the DM. Partially due to this reason, it is difficult to quantitatively evaluate the quality of preferred solutions obtained by various preference-based EMO algorithms under a unified framework.

In addition, although we criticised the ineffectiveness of *a posteriori* decision-making process at the outset of this paper, there is no conclusive evidence to support the assertion that incorporating preference in EMO is always superior to the traditional EMO for approximating solution(s) of interest (SOI). In particular, since the search process of a preference-based EMO algorithm is usually restricted to a certain region tentatively towards the ROI, it has a risk of losing population diversity and end up converging to an unexpected region.

Bearing the above mentioned considerations in mind, this paper empirically investigates the effectiveness of different algorithms, including both preference- and non-preference-based ones, for approximating various SOI. In particular, we assume that the DM elicits her/his preference information as a reference point $\mathbf{z}^r = (z_1^r, \cdots, z_m^r)^T$ where each component represents the DM's expected value on that objective. To have a quantitative comparison, we use our recently developed R-metrics [15] to evaluate the quality of obtained SOI. In this paper, we aim to address the following five research questions (RQs) through our empirical studies.

*RQ*1 : *Is preference incorporation in EMO really superior to traditional EMO for approximating SOI?*

*RQ*2 : *What is the most effective way to utilise the preference information in EMO for approximating SOI?*

*RQ*3 : *How does the location of a reference point influence the performance of a preference-based EMO algorithm for approximating SOI?*

*RQ*4 : *If the DM's preference information is set in an interactive manner according to the evolution status, how does it influence the results?*

*RQ*5 : *Is that possible to use preference-based EMO algorithms to approximate the whole PF rather than merely a partial region?*

In the rest of this paper, Section 2 provides a pragmatic review of the current developments of preference-based EMO. Section 3 describes the methodologies that we used to set up the experiments, including algorithms, benchmark problems, different reference point settings, performance metrics. Section 4 presents and analyses the experimental results in accordance with the RQs. At the end, Section 5 concludes this paper and sheds some lights on future directions.

2 Literature Review

As introduced in Section 1, there is a growing trend of incorporating the DM's preference information into EMO [16–19] to approximate her/his preferred Pareto-optimal solutions in the past decades. Generally speaking, a preference-based EMO (PBEMO) process can be broken down into four essential components as shown in Fig. 1.

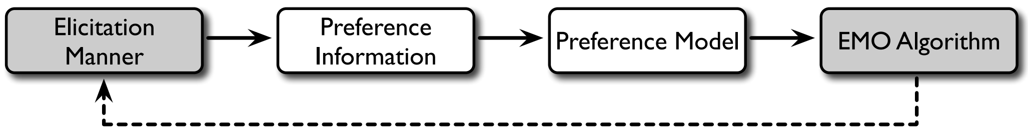

Figure 1: Flowchart of a PBEMO process.

- The elicitation manner decides when to ask the DM to elicit her/his preference information. There are three different elicitation manners: *a posteriori* (i.e., after a complete run of an EMO algorithm), *a priori* (i.e., before running an EMO algorithm) and *interactive* (i.e., during the running of an EMO algorithm).

- The preference information is the way of how does the DM express her/his preference. Perhaps the most straightforward one is a reference point, as known as an aspiration level vector, which represents the expected value the DM wants to achieve. The other one is through holistic comparisons which can be based on either the comparisons on solutions or objective functions. As for the prior one, it can be implemented as pairwise comparisons and a qualitative classification of solutions, etc. Whilst the comparison on objective functions can be implemented by: 1) assigning weights to different objectives, 2) a redefinition of trade-off relation or 3) a classification of objective functions.

- The preference model is the way how the preference information elicited by the DM can be used in an EMO algorithm. In the literature, VF, dominance relation and decision rules [20] are the most popular choices. In particular, a VF is a scalar function of all objectives which evaluates solutions quantitatively. Dominance relation describes the DM's preference in the form of the relation of a pair of solutions. Decision rules model the DM's preference as a set of '`IF-THEN`' rules.

- An EMO algorithm is the search engine that iteratively approximates SOI according to the preference model. In principle, any EMO framework (i.e., dominance- [8, 10, 11], indicator- [9] and decomposition-based frameworks [21–24]) can be used at this stage.

Note that the PBEMO process shown in Fig. 1 is a closed-loop system when using an *interactive* elicitation manner. Otherwise, it is a one-off process. In particular, the EMO algorithm is the starting point of this process when the elicitation manner is *a posteriori*. On the other hand, it is the ending point when *a priori* elicitation manner is used. In the following paragraphs, we will provide an overview on the current development of PBEMO mainly according to the elicitation manner, intertwined with the preference information and the preference model. Interested readers are also recommended to refer to two recent survey papers at this topic [25, 26].

2.1 A Priori Elicitation Manner

2.1.1 Using Reference Point as Preference Infromation

This is the most widely used way to express and model the DM's preference information. The first attempt along this line is from Fonseca and Fleming [27] who suggested to model the DM's preference as a goal that indicates desired levels of performance in each objective dimension. Afterwards, the reference point(s) were used in various ways to guide the EMO process towards the ROI. For example, in [11, 28] and [29], Deb et al. used the Euclidean distance to the reference point(s) as a second criterion (additional to the Pareto dominance) to evaluate the fitness of a solution. In particular, solutions closer to the reference point(s) have a higher priority to survive. Based on the similar merit, in [21] and [30], the reference point is used to help select the leader swarm in the multi-objective particle swarm optimisation algorithm. In [9], to consider DM's preference information, Thiele et al. made a simple modification on IBEA by incorporating the achievement scalarising function (ASF) into a binary indicator. In [31], Gong et al. proposed a set-based many-objective optimisation algorithm guided by a preferred region which is determined by reference point and ASF.

Furthermore, the relative position with respect to the reference point can be used to define a new dominance relation as well. For example, Molina et al. [32] suggested the g-dominance where solutions satisfying either all or none aspiration levels are preferred over those satisfying some aspiration levels. Said et al. [33] developed the r-dominance, where non-dominated solutions, according to the Pareto dominance relation, can be differentiated by their weighted Euclidean distances towards the reference point.

Instead of being directly used to guide solutions towards the ROI, the reference point can also be used to change the distribution of weight vectors, which are the core design components in the emerging decomposition-based EMO algorithms, according to the DM's preference information, e.g., [23, 34–36]. In [37], Narukawa et al. proposed an interesting preference-based NSGA-II where the DM's preference information is expressed as Gaussian functions on a hyperplane. In addition, reference points are also core components of $R2$ indicator, a set-based performance indicator. In [38] and [39], $R2$ indicator is modified to consider the DM's preference information.

Comparing to the other preference modelling tools, reference point is relatively intuitive to represent the DM's preference information. Without a demanding effort, the DM is able to guide the search towards the ROI directly or interactively even when encountering a large number of objectives. Recently, the first author and his collaborators developed a systematic way to evaluate and compare the performance of preference-based EMO algorithms using reference points for approximating the ROI [15]. This work lays the foundation to rigorously evaluate and compare different preference-based EMO algorithms by using reference point(s).

2.1.2 Using Weights as Preference Information

Its basic idea is to assign weights to different objectives according to their relative importance. For example, Deb [6] developed a modified fitness sharing mechanism, by using a weighted Euclidean distance, to bias the population distribution. Branke et al. [5] proposed a modified dominance principle where the trade-off among two objectives is directly specified by the DM, e.g., a gain/degradation of an objective by one unit will lead to a corresponding degradation/gain in the other objective. In [7], Branke and Deb developed a linearly weighted utility function that projects solutions to a hyperplane before evaluating the crowding distance in NSGA-II. In [40], Zitzler et al. showed how to use a weight distribution function on the objective space to incorporate preference information into Hypervolume-based EMO algorithms. Based on the same merit, Friedrich et al. [41] generalised this idea to two dominance-based EMO algorithms NSGA-II and SPEA2 [42].

It is worth noting that the weight-based methods become ineffective when facing a large number of objectives. Because it is difficult to neither specify the weights nor verify the quality of the biased approximation. Moreover, it is unintuitive and challenging for the DM to steer the search process towards the ROI via the weighting scheme. In addition, the weight-based methods are unable to approximate multiple ROIs and control the extent of the ROI.

2.1.3 Using Desirability Function (DF) as Preference Information

DF [43] aims to map each individual objective into a desirability with a value bounded within the range $[0, 1]$. Through this mapping, values of different objectives become comparable. Moreover, DFs are also able to prevent a biased distribution of solutions caused by badly scaled objectives. Afterwards, DFs are integrated with a popular indicator-based EMO algorithm, i.e., SMS-EMOA [44], to approximate the ROI. Note that the calculation of Hypervolume (HV) is based on the DFs instead of the original objective functions.

2.2 Interactive Elicitation Manner

In fact, almost all methods developed under *a priori* preference elicitation setting can be applied in an *interactive* manner. For example, the DM can periodically adjust the reference point to progressively guide the population towards the ROI. In the following paragraphs, we will overview some representative developments on the interactive MO.

2.2.1 Using Fuzzy Function as Preference Information

By classifying the relative importance of objectives into different grades, Cvetković and Parmee [45] developed a fuzzy preference relation that translates the pairwise comparisons among objectives into a weighted-dominance relation. In [46], Jin and Sendhoff developed a method to convert the DM's fuzzy preference information into weight intervals through pairwise comparisons on objectives. Shen et al. [47] proposed an interactive EMO algorithm based on fuzzy logic. In particular, after running the EMO algorithm for several generations, the DM is asked to specify the relative importance between pairs of objectives via linguistic terms. Thereafter, a new fitness function is defined according to a 'strength superior' relation derived from a fuzzy inference system.

2.2.2 Using Value Function (VF) as Preference Information

As a pioneer along this line, Phelps and Köksalan [13] proposed an interactive evolutionary meta-heuristic algorithm that translates the DM's pairwise comparisons of solutions into a linear programming problem. In particular, its optimal solution is the weights of an estimated VF in the form of a weighted sum whilst the estimated VF is used as the fitness function of the evolutionary meta-heuristic algorithm. In [48], Battiti and Passerini developed a progressively interactive EMO approach that uses learning-to-rank method to estimate the parameters of a polynomial VF. Afterwards, the derived VF is used to modify the Pareto dominance to compare solutions. In [14], Deb et al. developed an interactive EMO algorithm that progressively learns an approximated VF by asking the DM to compare a set of solutions in a pairwise manner. In [49], Branke et al. proposed to use robust ordinal regression to learn a representative additive monotonic VF compatible with the DM's preference information. Thereafter, the VF is used to replace the crowding distance calculation in NSGA-II. In [50], Pedro and Takahashi proposed to use a Kendall-tau distance to evaluate the accuracy of the approximated

VF learned by a radial basis function network. If the approximated VF is satisfactory, it is used to dynamically change the calculation of the crowding distance in NSGA-II to manipulate the density of solutions in a population. Instead of modelling the DM's preference information as VFs, Greco et al. [51] proposed to use decision rules to implement the preference modelling. In [52], Sun et al. proposed to build a surrogate model to approximate the DM's value function. In particular, they use a semi-supervised learning strategy to overcome the shortage of human labelled data during the interactive process.

In [53], Miettinen and Mäkelä developed an interactive multi-objective optimisation system called WWW-NIMBUS that allows the DM to classify objectives into up to five classes so as to find a more desirable solution. During the search process, the original MOP is transformed into a constrained single-objective optimisation problem by combining a weighted distance metric with an ASF. Later, Miettinen et al. [54] proposed the NAUTILUS method that starts from the nadir point and improves all objectives simultaneously in an interactive manner. In particular, the DM is able to specify either the frequency of interaction or the percentages of which (s)he would like to improve at each objective. Note that both WWW-NIMBUS and NAUTILUS use the classic mathematical programming techniques as the search engine. In [55], Sindhya et al. proposed to use EA to search for SOI under the framework of NAUTILUS.

In [56–58], Yang et al. proposed GRIST method that estimates the gradient of an underlying VF by using the indifference trade-offs provided by the DM in an interactive manner. Thereafter, the gradient is projected onto the tangent hyperplane of the PF so that the search process can be guided towards the direction along which the DM's utility can be improved. Recently, Chen et al. [59] applied the GRIST method in the context of EA to improve the versatility of the GRIST method for solving problems without nice mathematical properties such as convexity and differentiability.

2.2.3 Using Holistic Comparisons as Preference Information

Asking the DM to periodically select the most preferred solution from a set of candidates is another alternative way to represent the DM's preference information. For example, Folwer et al. [60] proposed to use the best and the worst solutions specified by the DM to construct convex preference cones. Thereafter, a cone dominance relation is defined to rank the population. In [61], Sinha et al. proposed a progressively interactive EMO algorithm that asks the DM to select the most preferred solution from an archive. The collected preference information is used to build polyhedral cones for modifying Pareto dominance relation. It is not uncommon that interaction with DM is full of uncertainty. In [62], Gong et al. proposed a preference-based EMO algorithm that treats the uncertainty as interval parameters where a preference polyhedron, constructed by convex cones, is used to approximate DM's preference information. Interested readers are referred to some more relevant works on interval multi-objective optimisation [63, 64]. In [65], Köksalan and Karahan proposed an interactive version of territory defining EA [66] to consider the DM's preference information in the loop. In particular, a territory is defined around each individual and the favourable weights of the best solution selected by the DM are identified to determine a new preferred weight region. In [67], Guo et al. proposed an interactive preference articulation method based on a three-step process, i.e., partitioning-updating-tracking, where the quality of a solution is measured according to the satisfaction of the semantic-based relative importance of different objectives.

In [22], Gong et al. proposed an interactive MOEA/D where the weight vector of the selected best solution is used to renew the preferred weight region. In particular, this region is a hypersphere with the preferred weight vector being the centre. Recently, the first author and his collaborators [24] proposed a systematic framework for incorporating the DM's preference information into the decomposition-based EMO algorithms. More specifically, it periodically asks the DM to score a couple of selected solutions according to their satisfaction to the DM's preference information. Based on the scoring results, a radial basis function network is trained to predict the fitness of solutions in the next several generations. Moreover, the fitness of solutions directly represent the priority of weight vectors. In other words, the best solution is associated with the most promising weight vector, so on and so forth. The other weight vectors are moved towards those selected promising weight vectors to represent the DM's preference information.

2.3 A Posteriori Elicitation Manner

In the *a posteriori* scenario, the DM has no chance to modify the existing trade-off alternatives obtained by an EMO algorithm. Instead, the *a posteriori* methods mainly aim to shortlist solutions that might be interested by the DM to support the decision-making process. The most popular one is to identify the knee points of which a small improvement in one objective can lead to a large deterioration in other objectives [68]. For example, Bhattacharjee et al. [69] developed a method that recursively uses the expected marginal utility measure to identify the SOI. Moreover, this method is also able to characterise the nature of those selected solutions (either internal or peripheral) through a set of systematically generated reference directions. Besides knee points, solutions lying on the edge of the approximated PF is useful for the DM to understand some important characteristics of the PF, e.g., its shape and boundary. In [70], Everson et al. proposed four definitions of edge points and examined their relations under a many-objective setting.

Different from the knee and edge points, subset selection is another alternative to find a pre-specified number of solutions that best represent the characteristics of the original PF. To this end, researchers (e.g., [71] and [72]) mainly aim to efficiently choose a limited number of representative solutions that achieve a minimisation of inverted generational distance (IGD) [73] or a maximisation of HV [74].

3 Experimental Setup

This section introduces the setup of our experiments, including the basic mechanisms of the selected traditional and preference-based EMO algorithms; the characteristics of the benchmark problems; the settings of reference points that represent various DM's preference information; and the performance metrics used to evaluate the quality of solution sets for approximating the ROI.

3.1 Peer Algorithms

It is well known that there are three major frameworks (i.e., dominance-, indicator- and decomposition-based frameworks) in the EMO literature. To study the effectiveness of preference incorporation in EMO, peer algorithms are chosen in accordance to this categorisation. In particular, we choose three iconic EMO algorithms, i.e., NSGA-III [75], IBEA [2] and MOEA/D [3] without considering the DM's preference information. Note that all of them are scalable to handle problems with more than three objectives. In addition, we choose six widely used preference-based EMO algorithms, i.e., g-NSGA-II [8], r-NSGA-II [33], R-NSGA-II [28], PBEA [9], RMEAD2 [34] and MOEA/D-NUMS [23] in our experiments. Note that all preference-based EMO algorithms use reference point(s) to represent the DM's preference information. Their differences mainly lie in the way of how to utilise the preference information to drive the search process. The following paragraphs briefly introduce the mechanisms of these selected peer algorithms whilst interested readers can find more details from their original papers. In addition, their corresponding parameter settings can be found in Section 1 of the supplementary document.

3.1.1 Traditional EMO Algorithms

- NSGA-III: it is an extension of NSGA-II where the mixed population of parents and offspring is first divided into several non-dominated fronts by using the fast non-dominated sorting procedure. Afterwards, solutions in the first several fronts have a higher priority to survive to the next generation. In particular, the exceeded solutions are trimmed according to the local density of a subregion specified by one of the evenly sampled weight vectors.

- IBEA: it transfers an MOP into a single-objective optimisation problem that optimises a binary performance indicator. In particular, this paper uses the binary additive ϵ-indicator defined as:

$$I_{\epsilon^+}(A,B) = \min_{\epsilon}\Big\{\forall \mathbf{x}^2 \in B, \exists \mathbf{x}^1 \in A : f_i(\mathbf{x}^1) - \epsilon \\ \leq f_i(\mathbf{x}^2), i \in \{1, \cdots, m\}\Big\}. \tag{2}$$

Then, this indicator is directly used to assign the fitness value to a solution $\mathbf{x}$ in the current population P:

$$F(\mathbf{x}) = \sum_{\mathbf{x}' \in P \setminus \{\mathbf{x}\}} -e^{-I_{\epsilon^+}(\{\mathbf{x}'\},\{\mathbf{x}\})/\kappa}. \tag{3}$$

- MOEA/D: its basic idea is to decompose the original MOP into several subproblems, either as a single-objective scalarising function or a simplified MOP. Then, a population-based technique is used to solve these subproblems in a collaborative manner. In particular, this paper chooses the widely used inverted Tchebycheff function as the subproblm formulation:

$$\begin{array}{ll} \text{minimise} & g^{tch}(\mathbf{x}|\mathbf{w},\mathbf{z}^*) = \max_{i=1,\cdots,m} \{|f_i(\mathbf{x}) - z_i^*|/w_i\} \\ \text{subject to} & \mathbf{x} \in \mathbf{\Omega} \end{array}. \tag{4}$$

Note that we do not allow $w_i = 0$ in setting $\mathbf{w}$ where replace $w_i = 0$ by $w_i = 10^{-6}$ in equation (4).

3.1.2 Preference-based EMO Algorithms

- R-NSGA-II: it uses the weighted distance between a solution $\mathbf{x}$ (belonging to the last acceptable non-dominated front) and $\mathbf{z}^r$ to replace the crowding distance of NSGA-II. In particular, the weighted distance is calculated as:

$$Dist(\mathbf{x},\mathbf{z}^r) = \sqrt{\sum_{i=1}^{m} w_i \left(\frac{f_i(\mathbf{x}) - z_i^r}{f_i^{\max} - f_i^{\min}} \right)^2}, \tag{5}$$

where $\sum_{i=1}^{m} w_i = 1$ and $w_i \in [0,1]$. $f_i^{\max}$ and $f_i^{\min}$ are respectively the maximum and minimum at the i-th objective. Furthermore, R-NSGA-II uses an ϵ-clearing strategy to avoid overcrowdedness within a local niche.

- r-NSGA-II: it defines a new dominance relation, called r-dominance, to incorporate the DM's preference information in NSGA-II. Specifically, given two solutions $\mathbf{x}^1$ and $\mathbf{x}^2$, $\mathbf{x}^1$ is said to r-dominate $\mathbf{x}^2$ if $\mathbf{x}^1$ dominates $\mathbf{x}^2$; or $\mathbf{x}^1$ and $\mathbf{x}^2$ are non-dominated according to the Pareto dominance, but $\overline{Dist}(\mathbf{x}^1,\mathbf{x}^2,\mathbf{z}^r) < -\delta$, where:

$$\overline{Dist}(\mathbf{x}^1,\mathbf{x}^2,\mathbf{z}^r) = \frac{Dist(\mathbf{x}^1,\mathbf{z}^r) - Dist(\mathbf{x}^2,\mathbf{z}^r)}{Dist_{\max} - Dist_{\min}}, \tag{6}$$

where $Dist_{\max}$ and $Dist_{\min}$ are respectively the maximum and minimum of $Dist(\mathbf{x},\mathbf{z}^r)$ in the current population. $\delta \in [0,1]$ is used to control the extent of the approximated ROI.

- g-NSGA-II: it defines a new dominance relationship called g-dominance in NSGA-II. Given $\mathbf{z}^r$, solutions dominated by or dominate $\mathbf{z}^r$ are more preferable than those non-dominated ones.

- PBEA: it integrates the DM's preference information into IBEA by modifying its additive ϵ-indicator as follows:

$$I_p(\mathbf{x}^1,\mathbf{x}^2) = I_\epsilon(\mathbf{x}^1,\mathbf{x}^2)/(s(\mathbf{x}^1) + \sigma - \min_{\mathbf{x}^2 \in P}[s(\mathbf{x}^2)]), \tag{7}$$

where P is the current population, $\sigma > 0$ controls the importance of different solutions with respect to $\mathbf{z}^r$. In particular, the smaller the σ is, the more solutions near $\mathbf{z}^r$ are favoured, $s(\mathbf{x})$ is the augmented Tchebycheff ASF:

$$s(\mathbf{x}) = \max_{i=1,\cdots,m} w_i(f_i(\mathbf{x}) - z_i^r) + \rho \sum_{i=1}^{m} (f_i(\mathbf{x}) - z_i^r), \tag{8}$$

ρ is a small augmentation coefficient.

- RMEAD2: it is a variant of MOEA/D where the DM's preference information is used to generate a set of weight vectors biased towards the DM supplied reference point. To this end, it gradually re-samples new weight vectors, according to a uniform distribution, in the vicinity of the solution with respect to the weight vector closest to $\mathbf{z}^r$.
- MOEA/D-NUMS: it uses a closed-form non-uniform mapping scheme to transform the originally evenly distributed weight vectors on a canonical simplex into new positions close to $\mathbf{z}^r$. Thereafter, the transformed weight vectors are used in MOEA/D or any other decomposition-based EMO algorithm to steer the search process towards the ROI either directly or interactively.

3.2 Test Problems

In this paper, we consider test problems chosen from six widely used benchmark suites including ZDT1 to ZDT6 [76], DTLZ1 to DTLZ6 [77], minus DTLZ1 to minus DTLZ4 [78], mDTLZ1 to mDTLZ4 [79], WFG3 [80] and correlated m-500 knapsack problems [81]. All these test problems, except the correlated knapsack problems, are with continuous variables and have various PF shapes (e.g., linear, convex, concave, disconnected, degenerate and inverted PFs) and different search space properties. ZDT problems have only two objectives. mDTLZ problems are with three objectives whilst the number of objectives of the other test problems is set to 3, 5, 8 and 10 respectively. The number of variables are set as recommended in their original papers.

3.3 Settings of Reference Points

In our experiments, we consider two types of reference point settings. One is called a 'balanced' setting where the reference point is placed at the centre region of the PF; whilst the other is called an 'extreme' setting where the reference point is placed at the vicinity of an extreme of the PF. For each case, we set three reference points, i.e., $\mathbf{z}^r_\mathtt{P}$ on the PF, $\mathbf{z}^r_\mathtt{I}$ in the infeasible region and $\mathbf{z}^r_\mathtt{F}$ in the feasible region. The detailed settings of reference points used in our experiments can be found in Section 2 of the supplementary document.

3.4 Preference Metrics

To have a quantitative comparison, we consider two levels of assessments. The first one is about the approximation accuracy. Given the DM supplied reference point $\mathbf{z}^r$, the approximation accuracy achieved by a solution set P is evaluated as:

$$\mathbb{E}(P) = \min_{\mathbf{x}\in P}\Big\{\max_{i=1,\cdots,m}(f_i(\mathbf{x}) - z_i^r)/w_i\Big\}, \tag{9}$$

The smaller the $\mathbb{E}(P)$ is, the better P is for approximating the DM most preferred solution. In particular, we set $w_i = \frac{1}{m}$ in our experiments given that all objectives are assumed to be of an equal importance.

The second assessment is based on our recently proposed R-metrics [15] including R-IGD and R-HV metrics. They are used to evaluate the quality of an approximation set for approximating the ROI with respect to the DM supplied reference point. The basic idea of R-metric evaluation is to pre-process the approximation sets found by different algorithms before using the IGD and HV for performance assessment. More details related to the R-metric calculation can be found in Section 3 of the supplementary document.

In the experiments, each algorithm is performed 31 independent runs. We keep a record of the median and the interquartile range of metric values obtained for different test problems with various reference point settings. The corresponding data are gathered in Tables 8 to 78 in Section 3 of the supplementary document. In particular, the best metric values are highlighted in bold face with a grey background. To have a statistically sound conclusion, we use the Wilcoxon signed-rank test at a 0.05 significance level to validate the statistical significance of the best median metric values. In addition, we keep a record of the ranks, with respect to the performance metrics, obtained by different algorithms on each test problem instance. These ranking data will be used as the building block to carry out our discussions upon the empirical results.

4 Empirical Results and Analysis

Due to the massive amount of data collected in our experiments, it will be messy if we pour all results in this paper. Instead, it is more plausible that we focus on some important observations contingent upon the RQs posed in Section 1. Interested readers can refer to Sections 4 to 7 of the supplementary document for the complete results including tables of metric values obtained by different algorithms, violin plots of ranks, along with the plots of population distribution.

4.1 Performance Comparisons of Preference and Non-Preference-Based EMO Algorithms

Let us start our discussion from the relatively simple 2-objective ZDT problems. To have a collective comparison among different algorithms, we use bar graphs, as shown in Fig. 2, to present the average ranks of $\mathbb{EP}(P)$, R-HV, R-IGD obtained by different algorithms with both 'balanced' and 'extreme' reference point settings[1]. From these results, we can see that the performance of NSGA-III, IBEA and MOEA/D is competitive comparing to those preference-based EMO algorithms. From the population plots shown in the supplementary document, we find that NSGA-III, IBEA and MOEA/D do not have any difficulty to approximate the whole PF. Therefore, they have a good chance to find SOI no matter where the reference point is placed. In contrast, all preference-based EMO algorithms considered in this paper were designed to approximate the ROI to which their approximated solutions have shown certain offsets. This explains the relatively inferior ranks of R-IGD and R-HV obtained by some preference-based EMO algorithms (e.g., r-NSGA-II, R-NSGA-II and RMEAD2) as shown in Fig. 2. On the other hand, if we consider the $\mathbb{E}(P)$ metric, the performance of those non-preference-based EMO algorithms are not as competitive as that on the R-IGD and R-HV. This observation can be attributed to the guidance provided by $\mathbf{z}^r$. Therefore, some preference-based EMO algorithms, can have a better approximation to the DM most preferred solution, i.e., the one closest to $\mathbf{z}^r$.

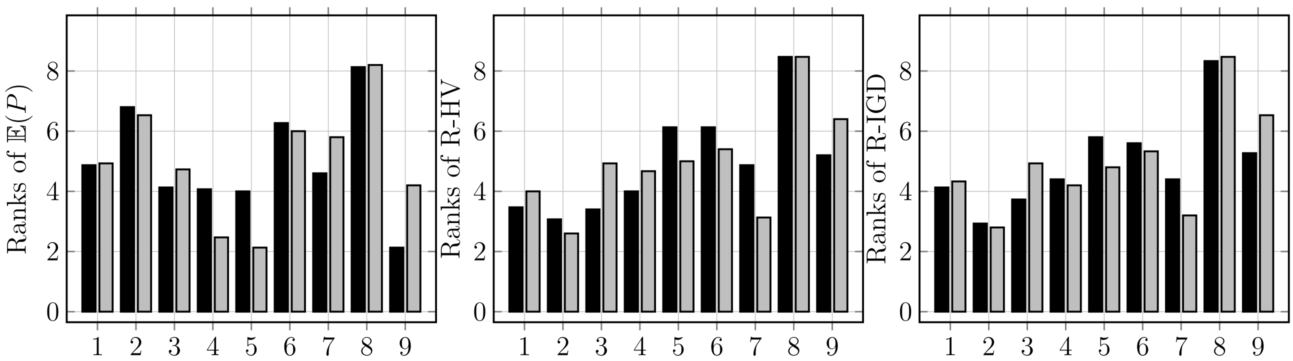

Figure 2: Bar graphs of the average ranks of $\mathbb{E}(P)$, R-HV, R-IGD obtained by different algorithms for 2-objective ZDT problems with both 'balanced' (black bars) and 'extreme' (gray bars) reference point settings.

Let us move to the DTLZ problems scalable to any number of objective. According to the results shown in Fig. 3, we find that the performance of non-preference-based EMO algorithms in the 3-objective case is not as competitive as the 2-objective scenario. In particular, IBEA can only find some special solutions (e.g., extreme points or boundary solutions) in many cases, as an example shown in Fig. 4. In contrast, the superiority of some preference-based EMO algorithms becomes more evident with the increase of the number of objectives. This can be explained as the expansion of the size of the PF with the dimensionality. In this case, solutions obtained by the non-preference-based EMO algorithms are sparsely distributed in a high-dimensional space (e.g., considering the example shown in Fig. 5). In other words, the chance for covering, by using a limited number of points, the expected ROI gradually decreases with the dimensionality. Moreover, it is widely accepted that solving a many-objective optimisation problem itself is very challenging.

[1]To save space, the x-axis of the bar graph uses indices to represent different algorithms, i.e., 1⇝NSGA-III, 2⇝IBEA, 3⇝MOEA/D, 4⇝g-NSGA-II, 5⇝r-NSGA-II, 6⇝R-NSGA-II, 7⇝PBEA, 8⇝RMEAD2, 9⇝MOEA/D-NUMS. This labelling rule applies to other bar graphs in this paper.

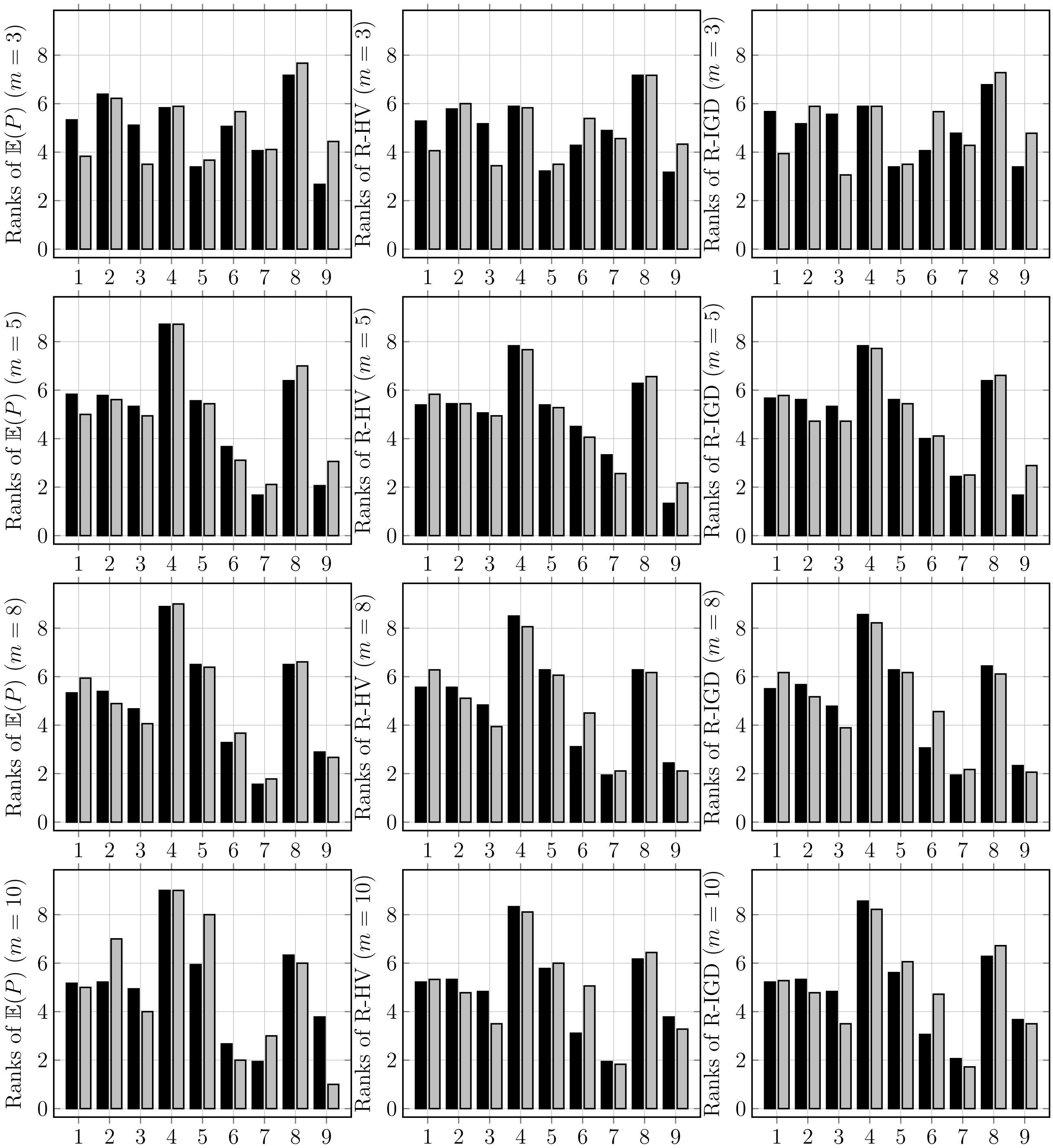

Figure 3: Bar graphs of the average ranks of $\mathbb{E}(P)$, R-HV, R-IGD obtained by different algorithms for 3, 5, 8 10-objective DTLZ problems with both 'balanced' (black bars) and 'extreme' (gray bars) reference point settings.

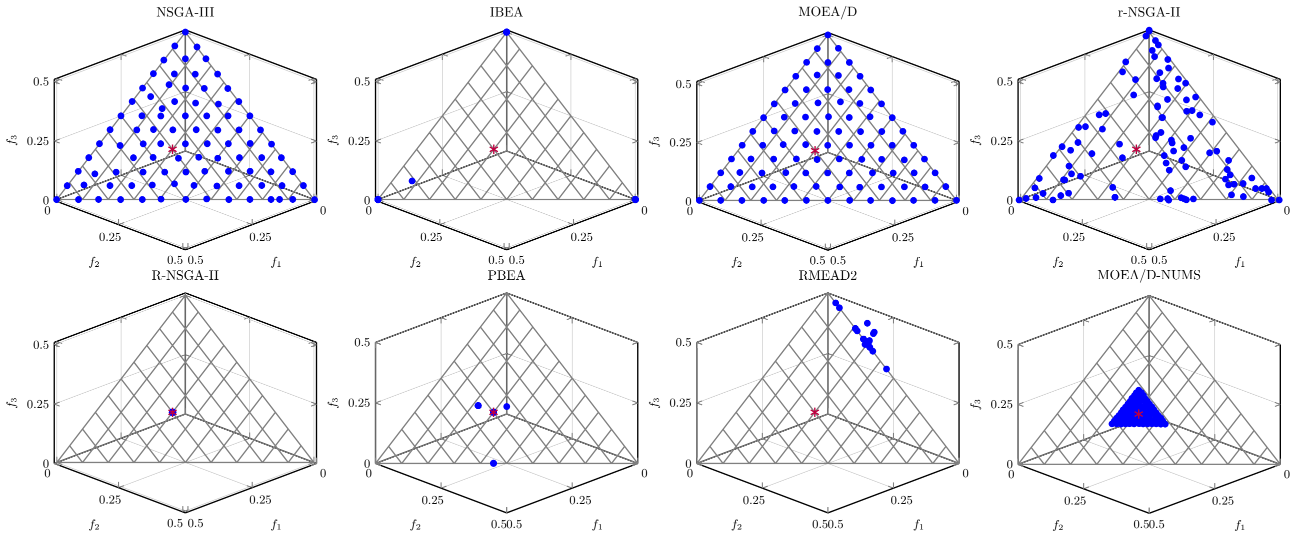

Figure 4: Plots of solutions obtained by different algorithms with the best R-IGD on 3-objective DTLZ1 when setting $\mathbf{z}_{\mathrm{P}}^{r} = (0.2, 0.15, 0.15)^T$.

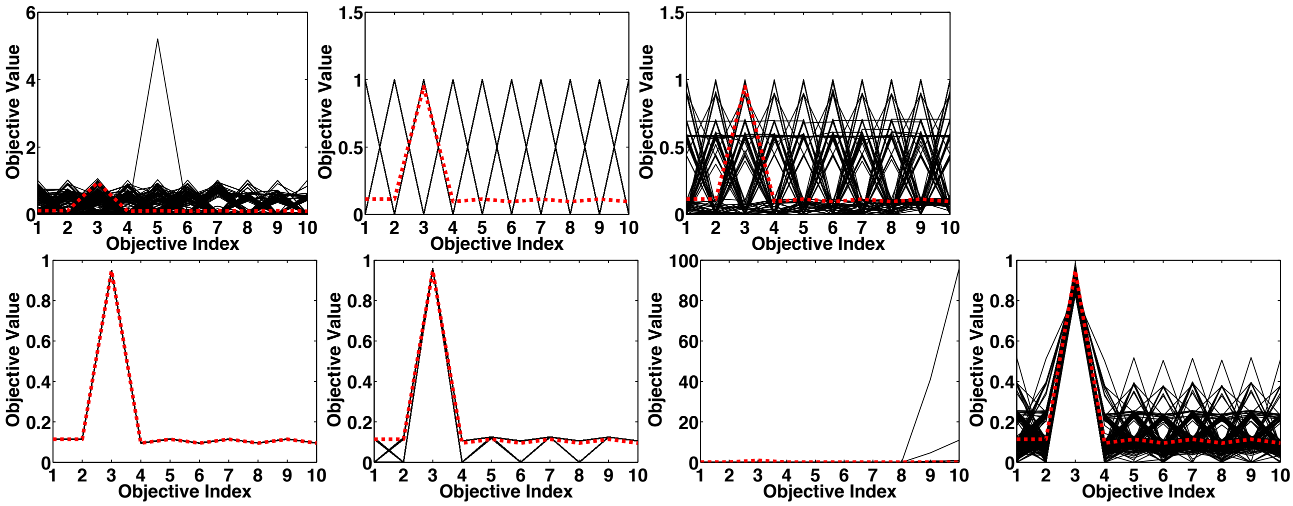

Figure 5: Parallel coordinate plots of solutions obtained by different algorithms with the best R-IGD on 10-objective DTLZ3 when setting $\mathbf{z}_{\mathrm{P}}^{r} = (0.114, 0.114, 0.948, 0.095, 0.114, 0.095, 0.114, 0.095, 0.114, 0.095)^T$.

Due to the regularly oriented PFs, the original DTLZ problems were criticised to be too simple to fully validate the behaviour of EMO algorithms, especially those decomposition-based ones [78]. Bearing this consideration in mind, Ishibuchi et al. [78] proposed minus DTLZ problems (denoted as DTLZ^{-1}) that have inverted PFs. From the bar graphs shown in Fig. 6, we can see that preference-based EMO algorithms, except RMEAD2 and MOEA/D-NUMS, have shown very promising performance across 3- to 10-objective cases. In particular, PBEA is consistently the best algorithm given that it is always able to find solutions focused in the ROI. As for RMEAD2 and MOEA/D-NUMS, due to the use of weight vectors as the driving force of the search process, their solutions are misled to a region away from the ROI (as shown in Fig. 7). Since the PFs of DTLZ^{-1} problems are not only inverted but also have different scales, the movements of weight vectors used in RMEAD2 and MOEA/D-NUMS are skewed to a wrong orientation. In contrast, similar to the observations on DTLZ problems, the performance of non-preference-based EMO algorithms is acceptable when $m = 3$ whereas they degenerate with the increase of m.

In addition to regularly oriented PFs, the original DTLZ problems were also criticised to just have one distance function which does not pose a sufficient convergence challenge [79]. Henceforth, Wang et al. [79] developed mDTLZ problems that can cause significant challenges to EMO algorithms by introducing hardly dominated boundaries. From the bar graphs shown in Fig. 8, we can see that PBEA and MOEA/D-NUMS are the most competitive algorithms for mDTLZ problems. According to the population plots shown in the supplementary document, we can see that both PBEA and MOEA/D-NUMS are always able to find well converged solutions lying in the ROI. Note that although the PFs of mDTLZ problems are inverted, they have the same scale at different objectives. In this case, MOEA/D-NUMS shows significantly better performance than that on DTLZ^{-1} problems. In contrast, the performance of the other four preference-based EMO algorithms is problem dependent. They can find acceptable SOI for mDTLZ2 and mDTLZ4 problems (as an example shown in Fig. 9) whereas all solutions are trapped in the hardly dominated boundaries on multi-modal functions mDTLZ1 and mDTLZ3 (as an example shown in Fig. 10). Since the mDTLZ problems were designed with only three objectives, it is less surprising to see the competitive performance of those non-preference-based EMO algorithms. Since NSGA-III, IBEA and MOEA/D are able to approximate the whole PF, they have a good chance to find SOI on mDTLZ2 and mDTLZ4 problems. Even for the difficult mDTLZ1 and mDTLZ3 problems, the solutions found by those non-preference-based EMO algorithms are closer to the PF than those found by the other four preference-based EMO algorithms.

WFG3 is an interesting test problem whose PF consists of both a degenerate and a non-degenerate parts [80]. To investigate the influence of a mixture of degenerate and non-degenerate PFs, we separately specify reference points targeting at these two parts. From the bar graphs shown in Fig. 11, we find that most preference-based EMO algorithms outperform those non-preference-based counterparts. In particular, PBEA and MOEA/D-NUMS are again the best algorithms on this test problem. As an example shown in Fig. 12, all preference-based EMO algorithms can find solutions lying round the ROI whilst those non-preference-based counterparts find too many irrelevant solutions across the whole PF. Note that solutions found by NSGA-III are entirely dominated by the other peers thus they were not plotted.

At the end, let us consider test problems with correlated objectives. One of the key characteristics of such problems is a reduced dimensionality of the PF. We use the correlated knapsack problems proposed by Ishibuchi et al. [81] in our experiments. Specifically, the original objective functions are transformed to $\mathbf{G}(\mathbf{x}) = (g_1(\mathbf{x}), \cdots, g_m(\mathbf{x}))^T$ as:

$$\begin{aligned} g_i(\mathbf{x}) &= f_i(\mathbf{x}), i = 1, 2, \\ g_i(\mathbf{x}) &= \alpha f_i(\mathbf{x}) + (1 - \alpha) f_1(\mathbf{x}), i = 2k + 1, \\ g_i(\mathbf{x}) &= \alpha f_i(\mathbf{x}) + (1 - \alpha) f_2(\mathbf{x}), i = 2k, \end{aligned} \tag{10}$$

where $k \in \{1, \cdots, \lceil \frac{m}{2} \rceil\}$ and $\alpha \in \{0.1, 0.9\}$ as recommended in [81]. In particular, $\alpha = 0.1$ means the other objectives are strongly correlated with the first two objectives whilst $\alpha = 0.9$ represents the opposite case. Since the PFs of the correlated knapsack problems are unknown a priori, we do not calculate the R-IGD metric in this case. From the bar graphs shown in Fig. 13, we can see that non-preference-based EMO algorithms, especially NSGA-III and IBEA, are the most competitive

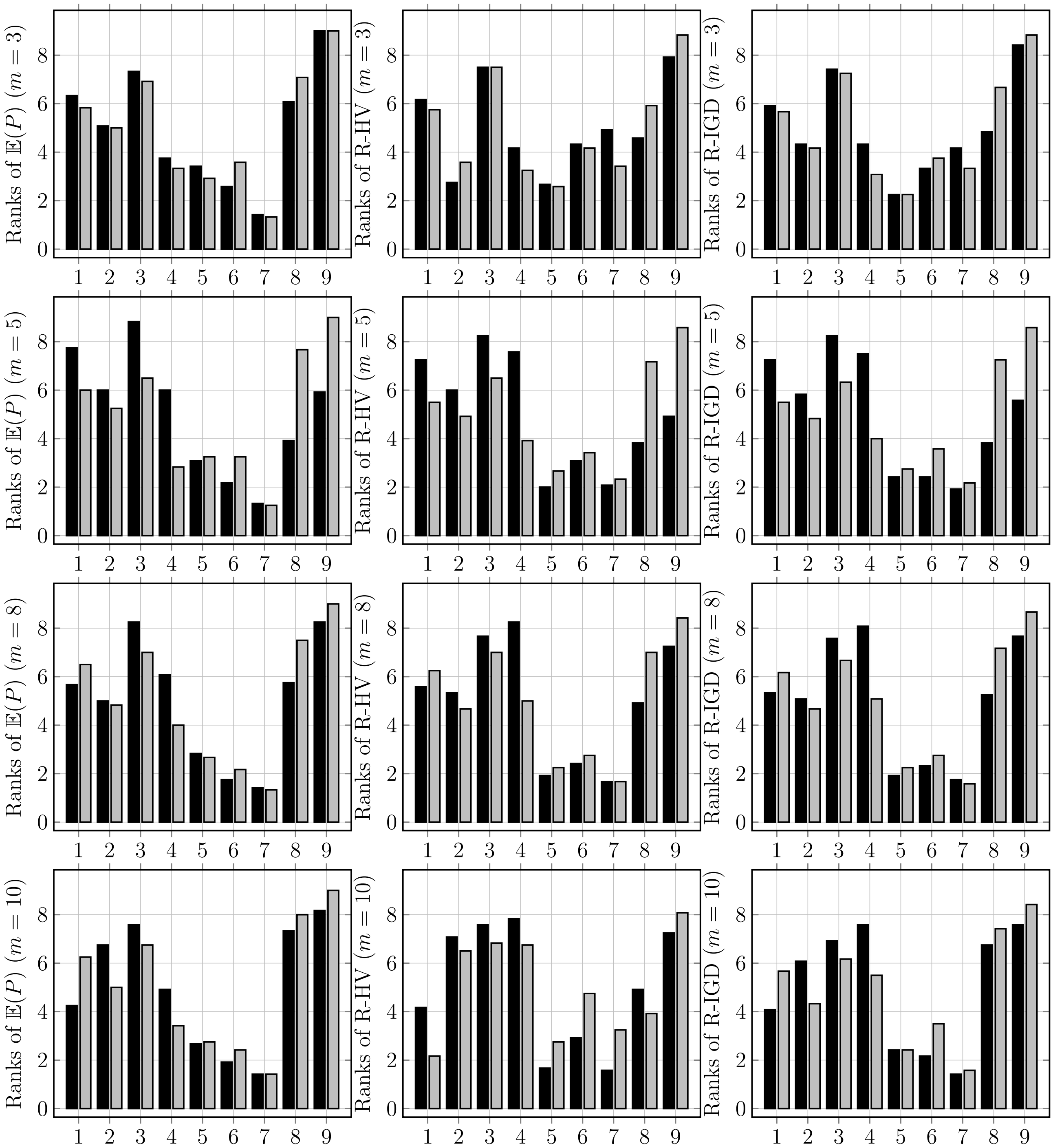

Figure 6: Bar graphs of the average ranks of $\mathbb{E}(P)$, R-HV, R-IGD obtained by different algorithms for 3, 5, 8 10-objective DTLZ^{-1} problems with both 'balanced' (black bars) and 'extreme' (gray bars) reference point settings.

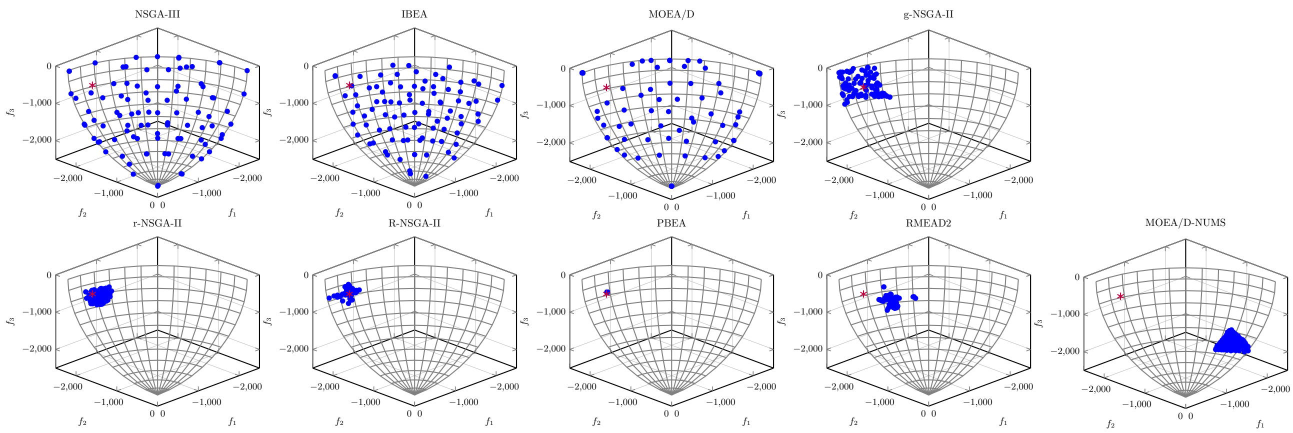

Figure 7: Plots of solutions obtained by different algorithms with the best R-IGD on 3-objective DTLZ3^{-1} when setting $\mathbf{z}_{\mathrm{P}}^{r} = (-664.4, -1993.1, -664.4)^T$.

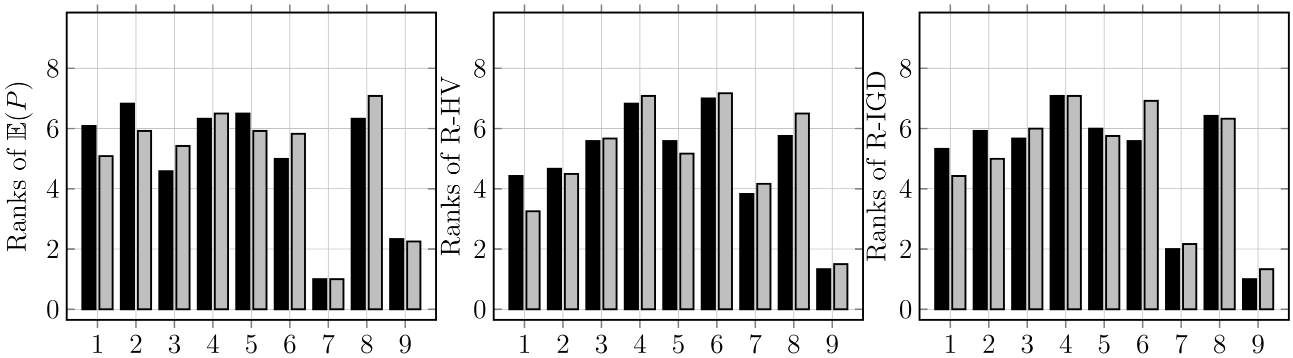

Figure 8: Bar graphs of the average ranks of $\mathbb{E}(P)$, R-HV, R-IGD obtained by different algorithms for 3-objective mDTLZ problems with both 'balanced' (black bars) and 'extreme' (gray bars) reference point settings.

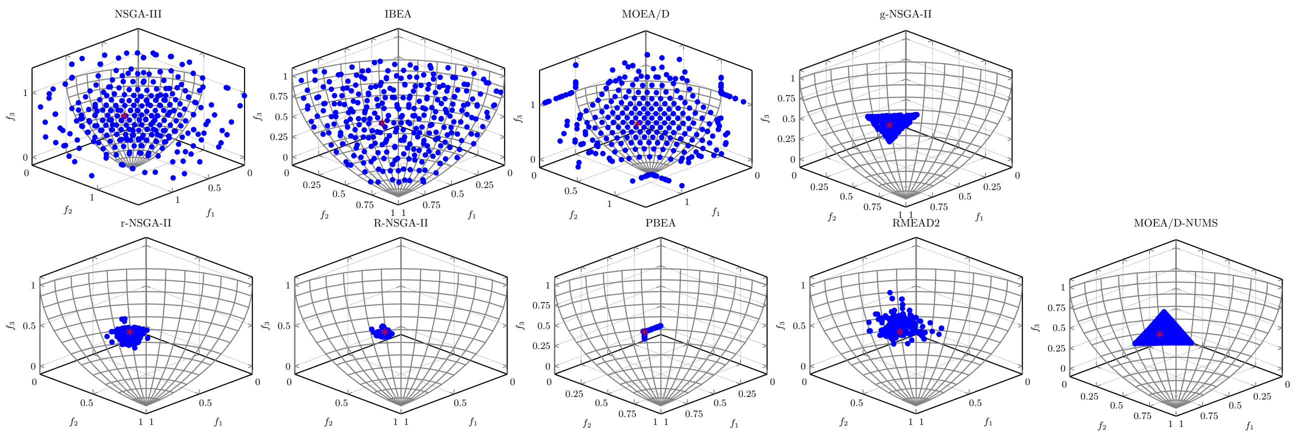

Figure 9: Plots of solutions obtained by different algorithms with the best R-IGD on 3-objective mDTLZ2 when setting $\mathbf{z}_{\mathrm{P}}^{r} = (-2.88, -2.16, -2.16)^T$.

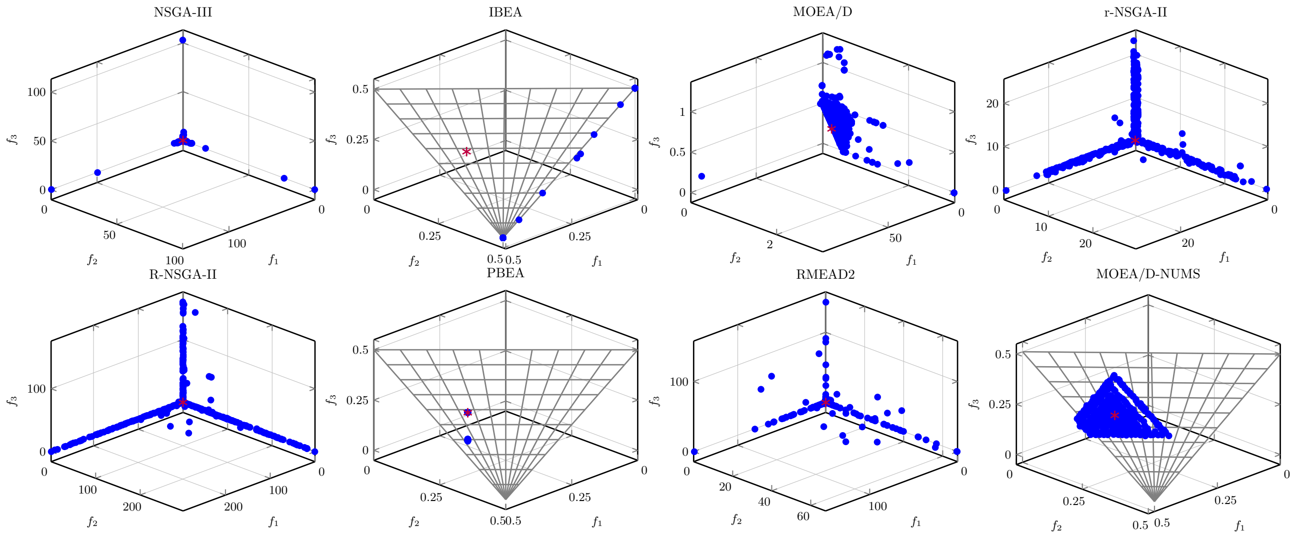

Figure 10: Plots of solutions obtained by different algorithms with the best R-IGD on 3-objective mDTLZ1 when setting $\mathbf{z}^r_{\mathsf{P}} = (-110, -110, -330)^T$.

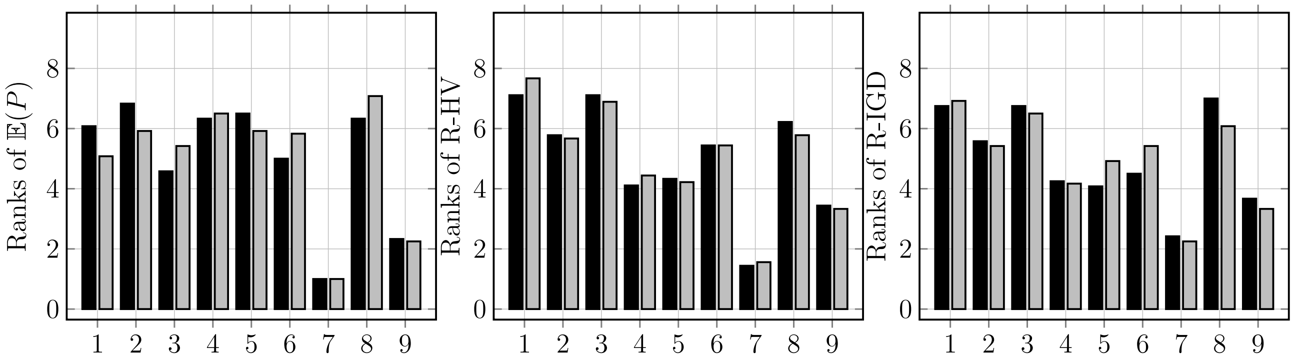

Figure 11: Bar graphs of the average ranks of $\mathbb{E}(P)$, R-HV, R-IGD obtained by different algorithms for 3, 5, 8 10-objective WFG3 problems with both ‘balanced’ (black bars) and ‘extreme’ (gray bars) reference point settings.

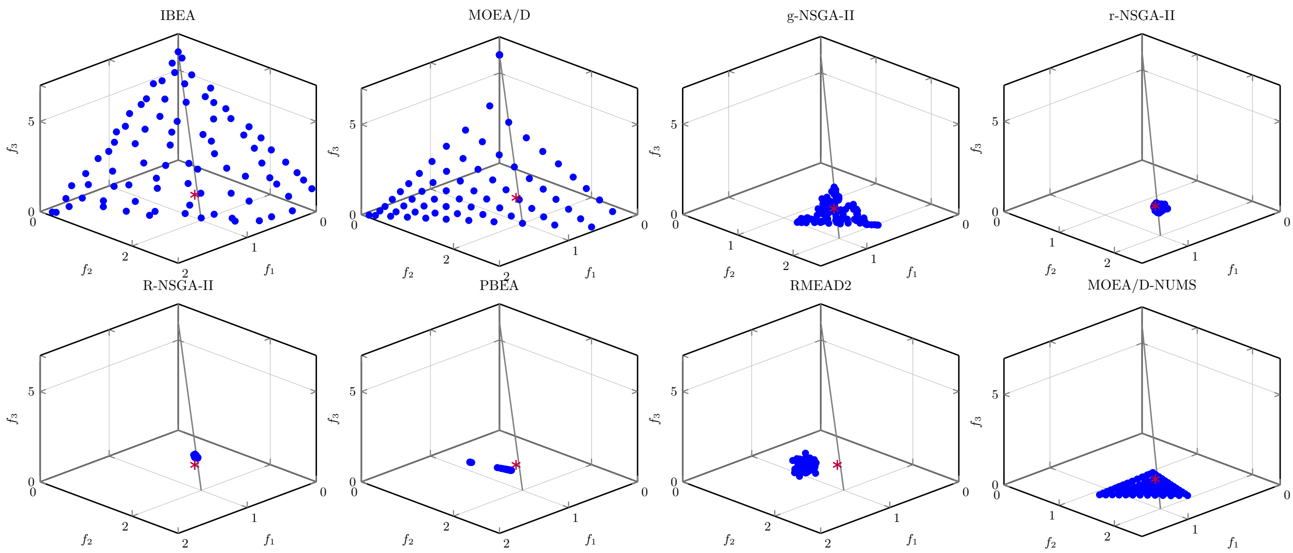

Figure 12: Plots of solutions obtained by different algorithms with the best R-IGD on 3-objective WFG3 when setting $\mathbf{z}^r_{\mathsf{P}} = (0.9, 1.8, 0.6)^T$.

algorithms when considering the R-HV metric. As shown in Fig. 14, we find that both NSGA-III and IBEA are able to approximate the entire PF whilst those solutions obtained MOEA/D are scattered at some focused regions. Although NSGA-III is also a decomposition-based EMO algorithm, its superior performance might be partially attributed to its normalisation operation. In contrast, PBEA and MOEA/D-NUMS, which have shown strong performance on the previous test problems, become almost the worst algorithms on correlated knapsack problems. For example, all solutions found by them and RMEAD2 are dominated by the other algorithms on the 3-objective strongly correlated knapsack problem as shown in Fig. 14. However, when considering the $\mathbb{E}(P)$ metric, some preference-based EMO algorithms, r-NSGA-II and R-NSGA-II in particular, become the most competitive algorithms. As the plots of population distribution shown in the supplementary document, we find that both r-NSGA-II and R-NSGA-II are able to approximate a partial front close to the supplied reference point. Thus, they have a larger chance to find the SOI by incorporating the DM's preference information into the search process. Furthermore, it is interesting to note that r-NSGA-II and R-NSGA-II are more likely to obtain a set of solutions approximate the whole PF on strongly correlated problems than those weakly correlated ones as shown in Fig. 14 and Fig. 15.

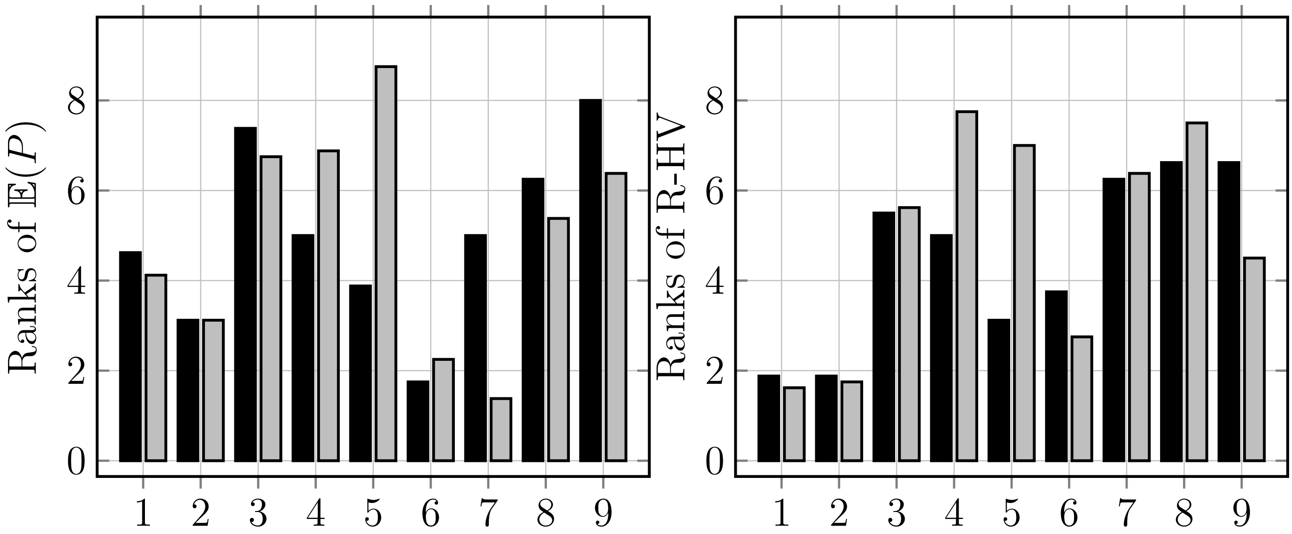

Figure 13: Bar graphs of the average ranks of $\mathbb{E}(P)$, R-HV obtained by different algorithms for 3, 5, 8 10-objective strongly (black bars) and weakly (grey bars) correlated knapsack problems.

Answers to $RQ1$: Incorporating preference information into an EMO algorithm does not always lead to a better approximation to the ROI comparing to those traditional EMO algorithms, especially when the number of objectives is small and problems with correlated objectives. However, with the increase of the number of objectives, incorporating preference information into the search process gradually becomes important. Due to the guidance provided by the DM supplied reference point(s), a preference-based EMO algorithm can have a better selection pressure towards the ROI. Furthermore, this is also beneficial to approximate the solution(s) most preferred by the DM, i.e., the one(s) closest to the DM supplied reference point.

4.2 Performance Comparisons of Different Preference-based EMO Algorithms

As discussed in Section 4.1, we appreciate the effectiveness of incorporating the DM's preference information for approximating the ROI. However, according to the results, we notice that not all preference-based EMO algorithms are able to have a desirable approximation to the ROI. In particular, some algorithms, where the DM's preference information is not appropriately utilised, were outperformed by non-preference-based EMO algorithms.

Let us first look into MOEA/D-NUMS whose performance is superior in many test problem instances. Because MOEA/D-NUMS considers the projection of the reference point on the simplex as one of the final biased weight vectors, it has a good chance to find the solution most preferred by the

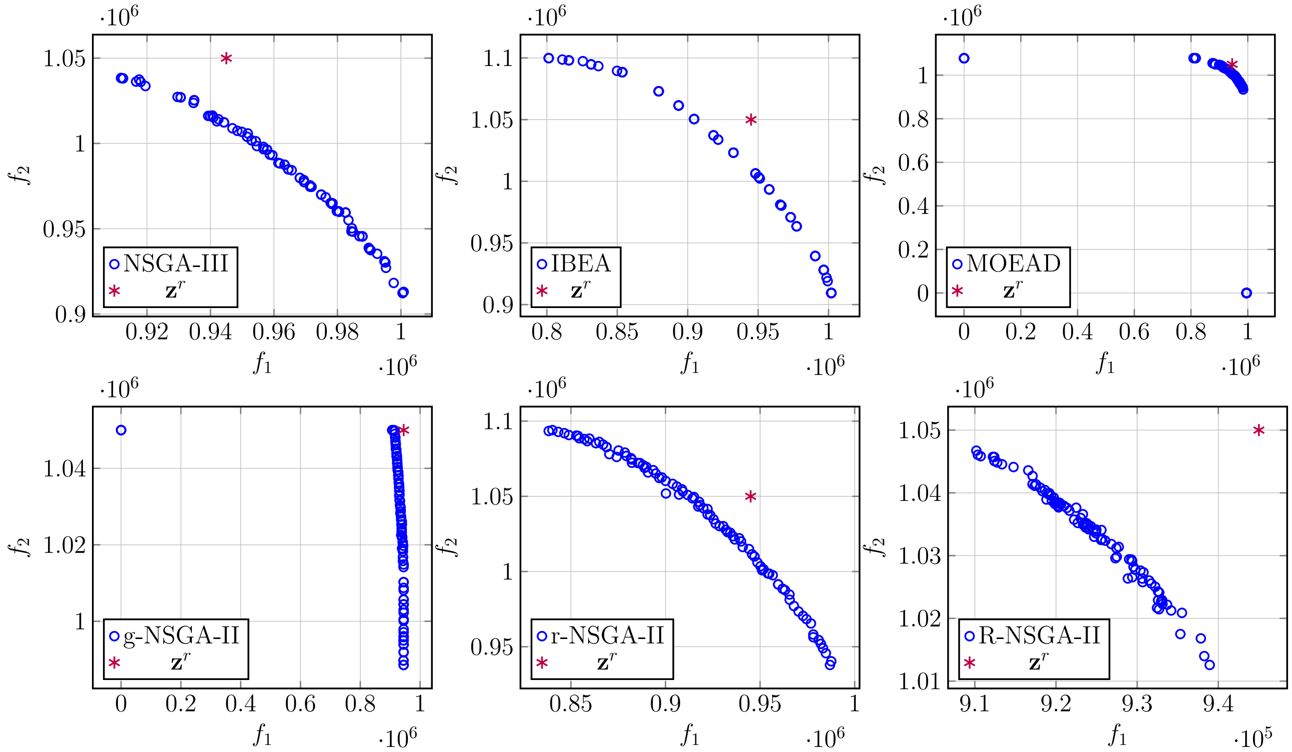

Figure 14: Plots of solutions obtained by different algorithms with the best R-IGD on 3-objective strongly correlated knapsack problem when setting $\mathbf{z}^r_{\mathrm{P}} = (-945, -1050, -945)^T \cdot 10^3$.

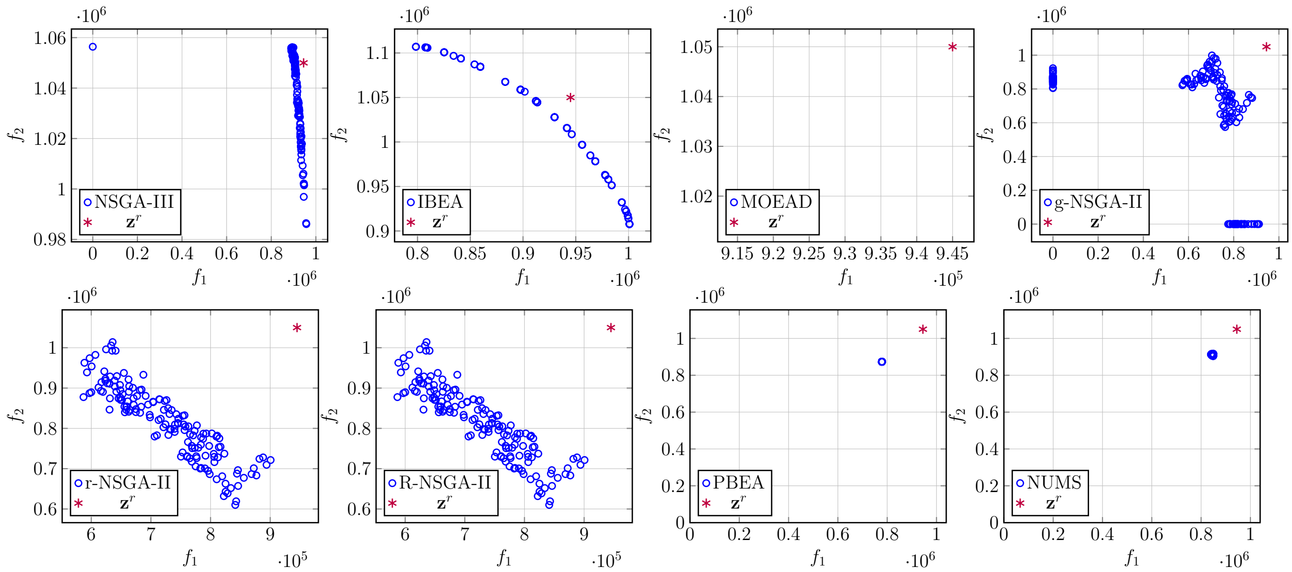

Figure 15: Plots of solutions obtained by different algorithms with the best R-IGD on 8-objective loosely correlated knapsack problem when setting $\mathbf{z}^r_{\mathrm{P}} = (-945, -1050, -945, -1050, -960, -1070, -975, -1080)^T \cdot 10^3$.

DM. Furthermore, it is well scalable to any number of objectives. However, we notice that MOEA/D-NUMS degenerates to be one of the worst algorithms on the minus DTLZ and correlated knapsack problems. As discussed in Section 4.1, its failure for minus DTLZ problems is attributed to: 1) an irregular PF shape that is significantly different from the canonical simplex; and 2) the disparately scaled objectives. As for the correlated knapsack problems, the failure of MOEA/D-NUMS can be attributed to the inability of its baseline MOEA/D [81]. Moreover, we notice that MOEA/D-NUMS can hardly find the extreme point(s) on the PF when the reference point is placed on one side of the PF. As the example shown in the left panel of Fig. 16, all weight vectors are shifted towards the projection of the reference point along the simplex, so that the extreme point is missed. Moreover, there is a tail extending towards the other end of the PF due to the non-uniform mapping. In contrast, the performance of RMEAD2, the other decomposition-based algorithm, is almost the worst preference-based EMO algorithm. Note that the weight vectors used in RMEAD2 gradually evolve towards the ROI with the population. Because the population evolution has some oscillations, it can be misleading to the adjustment of weight vectors. Moreover, as discussed in [82], frequently adjusting the distribution of weight vectors on-the-fly is negative to the search process of a decomposition-based EMO algorithm.

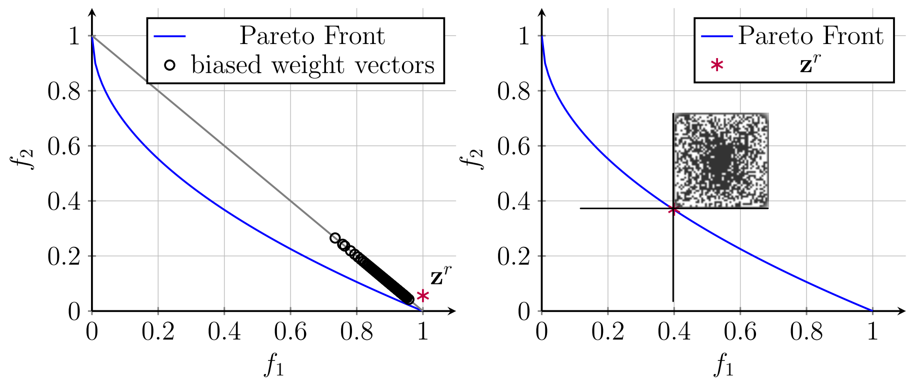

Figure 16: Illustrative examples of NUMS and g-dominance.

As for the three $*$-NSGA-II algorithms, their performance is similar in the 2-objective cases. Specifically, the selection pressure of g-NSGA-II comes from the box region specified by the DM supplied reference point. This is easy to implement in the 2-objective scenario. However, the effective area specified by the box region significantly decreases with the increase of the number of objectives. It can hardly provide sufficient selection pressure towards the ROI when the number of objectives is larger than two. This effect is similar to the original Pareto dominance, and it explains the inferior performance of g-NSGA-II in the 3- to 10-objective cases. Furthermore, it is worth noting that g-NSGA-II becomes ineffective when the reference point is set exactly on the PF. As shown in the right panel of Fig. 16, no solution will survive at the end except the DM supplied reference point. This effect has also been reflected by its poor performance when using a $\mathbf{z}^r_\mathbf{p}$ setting. The selection mechanisms of R-NSGA-II and r-NSGA-II are similar. Their major difference is: R-NSGA-II directly uses the Euclidean distance towards the DM supplied reference point to guide the selection; whilst r-NSGA-II has a parameter δ to control the comparability of two disparate non-dominated solutions. As a result, R-NSGA-II has shown more robust performance compared to r-NSGA-II. According to the plots of population distribution shown in the supplementary document, we notice that the extent of the approximated ROI obtained by either R-NSGA-II or r-NSGA-II is ad-hoc. There is no thumb-rule to set an appropriate parameter to control this extent. In addition, we find that r-NSGA-II and R-NSGA-II have shown competitive performance on problems with correlated objectives. This can be partially attributed to the superior performance of their baseline NSGA-II [81].

Different from the other preference-based EMO algorithms, which depend either on the Euclidean distance towards the reference point or a set of biased weight vectors, PBEA uses a Pareto-compliant indicator to assign fitness to each solution. In particular, the Pareto-compliant property is guaranteed by both the binary indicator and the ASF. According to the discussion in Section 4.1, we find that PBEA is very competitive across all scenarios, especially when the number of objectives becomes large, e.g., $m = 10$. Furthermore, PBEA is adaptable to various PF shapes and disparately scaled objectives. However, according to the plots of population distribution shown in the supplementary document, we also notice that the extent of the approximated ROI found by PBEA is very narrow. In fact, the width of the approximated ROI is controlled by the specificity parameter δ. But no thumb rule is available to set an appropriate δ for the desirable extent of the ROI. This is similar to R-NGSA-II and r-NSGA-II. In addition, similar to MOEA/D-NUMS, PBEA degenerates to one of the worst algorithms for correlated knapsack problems.

To have an overall picture of the performance comparison over different algorithms, we summarise the ranking results across all test instances and plot them as the heat map shown in Fig. 17. In particular, each cell of this heat map represents the number of times the corresponding algorithm has been ranked as a particular position in the performance comparison. For example, the cell $(1, 9) = 59$ indicates that MOEA/D-NUMS has been ranked as the first place for 59 times. From these comparison results, we find that some preference-based EMO algorithms (r-NSGA-II, R-NSGA-II, PBEA and MOEA/D-NUMS) have shown superior performance than those non-preference-based counterparts. In particular, PBEA is the most competitive algorithm as it is ranked as the top three positions in over 80% comparisons. This observation also supports the findings in the response to RQ1. On the other hand, we also find that some preference-based EMO algorithms (g-NSGA-II and RMEAD2) are ranked as the worst algorithms across all test instances.

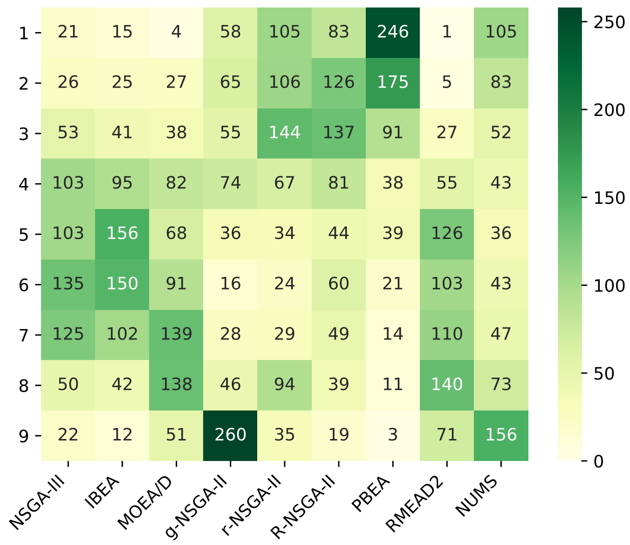

Figure 17: Heat maps of the number of times an algorithm has been ranked as a particular position in performance comparison.

<u>Answers to *RQ*2</u>: From our experiments, we find that r-NSGA-II, R-NSGA-II, PBEA and MOEA/D-NUMS are the most competitive preference-based EMO algorithms for approximating various SOI. Notice that these four algorithms are based on different EMO frameworks. Thus, we conclude that dominance-, indicator- and decomposition-based EMO frameworks are all useful for approximating the SOI, given the preference information supplied by the DM is well utilised. In particular, transforming preference information into a distance metric (e.g., Euclidean distance or Tchebycheff distance) is a reliable way to guide the search towards the SOI. However, we also appreciate the ineffectiveness of MOEA/D variants for problems with irregular PF shapes and disparately scaled objectives. On the other hand, an inappropriate use of the DM's preference information can even lead to a negative effect to the search process as analysed in Section 4.1. In particular, from our experiments, we can see that g-NSGA-II and RMEAD2 are worse than those non-preference based EMO algorithms in most cases. Last but not the least, almost all algorithms, except g-NSGA-II, claimed that the approximated ROI is controllable by some specific parameter(s). However, only MOEA/D-NUMS provides a tangible way to control the size of ROI; whilst the others are all set in an ad-hoc manner.

4.3 Influence of the Location of Reference Points

In the previous experiments, we find that preference-based EMO algorithms can have a decent approximation to the ROI if the DM supplied preference information is used in an appropriate manner. A natural question arises: *what happens if the DM supplies a 'bad' preference information that does not represent her/his actual aspiration?* In many real-world scenarios, it is not rare that the DM has little knowledge about the underlying black-box system at the outset of the optimisation process. Therefore, it is not trivial to set an appropriate reference point that perfectly represents the DM's preference information. In this subsection, we will investigate the influence of the setting of reference point, i.e., its location, on the performance of preference-based EMO algorithms. For proof of concept purpose and to facilitate a better visual understanding, here we only conduct experiments on 2- and 3-objective cases whilst the conclusions are able to be generalised to problems with a larger number of objectives according to our preliminary experiments.

Let us first look at two examples on ZDT1 where we consider two extreme reference point settings far away from the PF: $\mathbf{z}^{r_1} = (0.1, 0.1)^T$ in the infeasible region and $\mathbf{z}^{r_2} = (0.9, 0.9)^T$ in the feasible region. Fig. 18 plots the solutions obtained by six preference-based EMO algorithms with the best R-IGD values. From these results, we find that R-NSGA-II, PBEA and RMEAD2 work as usual. In particular, RMEAD2 normally cannot find well converged solutions. On the other hand, although the solutions obtained by r-NSGA-II and MOEA/D-NUMS well converge to the PF, they all show certain mismatch with respect to the ROIs. As for g-NSGA-II, its solutions almost cover the entire PF. As discussed in Section 4.2 and Fig. 16, the effective region of g-dominance is the box region covered by the DM supplied reference point. The farther the reference point away from the PF, the larger region covered by the reference point.

Let us look at another example on DTLZ2 with three objectives. Here we set the reference point as $\mathbf{z}^{r_3} = (-0.2, -0.2, -0.2)^T$ which dominates the ideal point. In particular, one may argue that the DM will not set negative values as a reference point. In this case, we assume that such reference point setting represents that the DM expects for solutions having a as good objective value as possible at each objective. From the experimental results shown in Fig. 19, we can see that almost all algorithms, except MOEA/D-NUSM, have shown some unexpected behaviour. Specifically, g-NSGA-II and PBEA almost degenerate to their non-preference-based baseline EMO algorithm, i.e., NSGA-II and IBEA, as the obtained solutions tend to cover the entire PF. r-NSGA-II, R-NSGA-II and RMEAD2 end up with solutions lying on a boundary of the PF in an add-hoc manner.

Answers to *RQ*3: From our experiments, we find that a preference-based EMO algorithm may not work as expected given a 'bad' reference point. In particular, a so called 'bad' choice is typically a reference point way beyond the PF. In this case, the DM supplied reference point is either far away the optima (s)he actually expects or too utopian to approach. In real-world black-box optimisation scenarios, it is not rare that the DM struggles to set a reasonably good reference point given her/his little knowledge about the underlying problem. This becomes even severer when having a large number of objectives.

4.4 Incorporating User Preference in an Interactive Manner

The previous experiments are conducted under the *a priori* elicitation manner. As discussed in Section 2.2, using a reference point to represent the DM's preference information can be directly used in an *interactive* manner. Different from many studies on preference-based EMO in the literature (e.g. [8,9,23,28,33,34]), which are mainly tested on benchmark problems, this paper considers testing the effectiveness of interactive EMO on stock market portfolio optimisation under a real-world setting. In particular, we collect the stock market data of 58 listed companies from Shenzhen Stock Exchange A Share since 1990. Two popular portfolio optimisation models are considered in this paper.

The first one is the Mean-Variance-Skewness (MVS) model proposed by Konno and Suzuki [83]. Specifically, given a portfolio of financial assets $\mathbf{P} = (\rho_1, \cdots, \rho_n)^T$ where ρ_i indicates the percentage of the wealth invested in the i-th asset and $\sum_{i=1}^{n} \rho_i = 1$, the return of $\mathbf{P}$ is calculated as:

$$\psi[\mathbf{P}] = \sum_{i=1}^{n} \rho_i r_i, \tag{11}$$

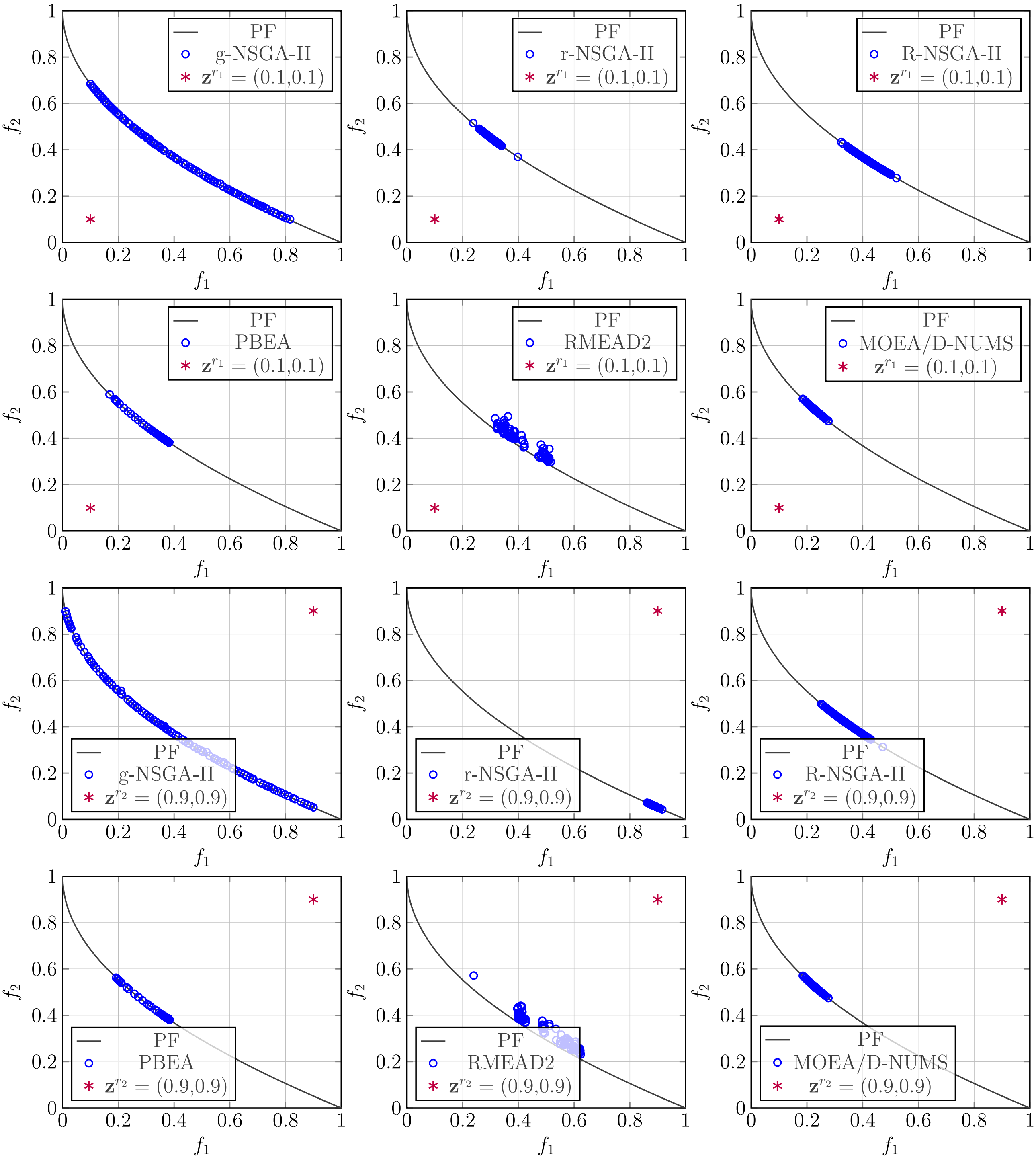

Figure 18: Solutions obtained by six preference-based EMO algorithms on the ZDT1 test problem when setting $\mathbf{z}^{r_1} = (0.1, 0.1)^T$ and $\mathbf{z}^{r_2} = (0.9, 0.9)^T$.

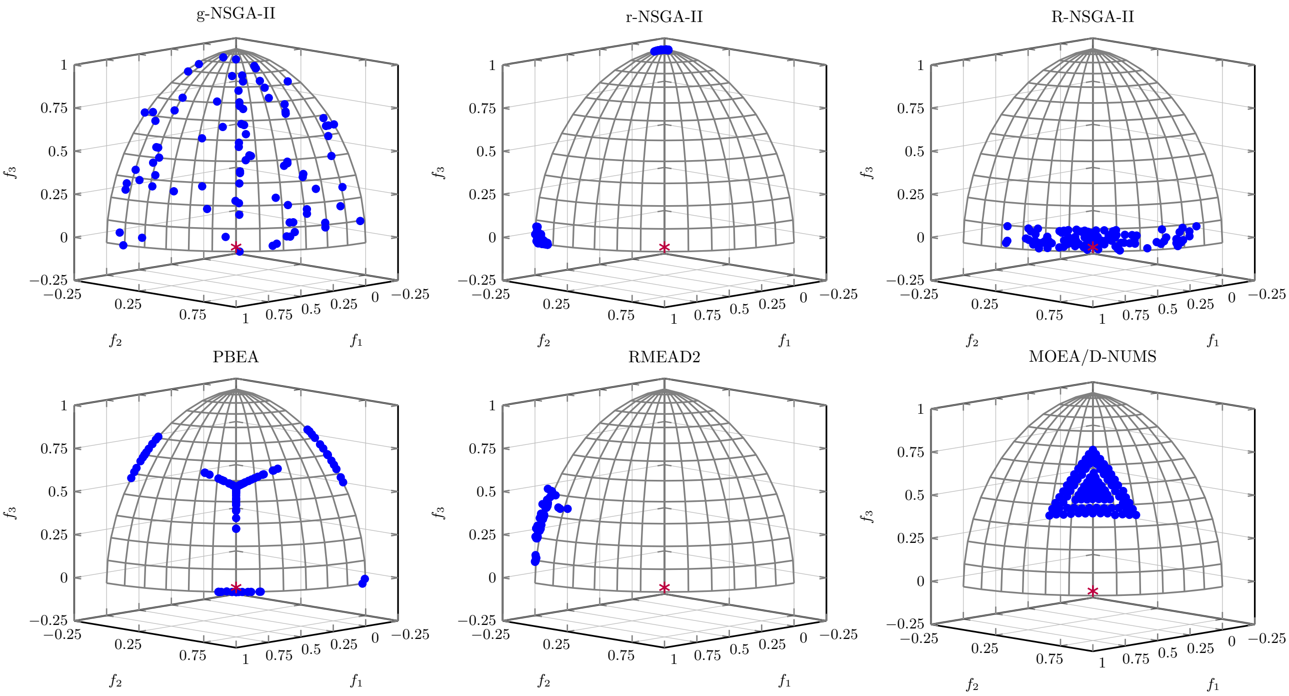

Figure 19: Solutions obtained by six preference-based EMO algorithms on the DTLZ2 problem where $\mathbf{z}^{r_3} = (-0.2, -0.2, -0.2)^T$.

where r_i is the rate of return of ρ_i, $i \in \{1, \cdots, n\}$. The MVS model is formulated as:

$$\begin{cases} \text{maximise} & \mathbb{E}[\psi(\mathbf{P})] = \sum_{i=1}^{n} \rho_i \mathbb{E}[r_i] \\ \text{minimise} & \mathbb{V}[\psi(\mathbf{P})] = \mathbb{E}[(\psi(\mathbf{P}) - \mathbb{E}[\psi(\mathbf{P})])^2] \\ \text{maximise} & \mathbb{S}[\psi(\mathbf{P})] = \mathbb{E}[(\psi(\mathbf{P}) - \mathbb{E}[\psi(\mathbf{P})])^3] \end{cases}, \tag{12}$$

In the experiments, only R-NSGA-II, PBEA and MOEA/D-NUMS are chosen as the preference-based EMO algorithms given their superior performance reported in Section 4.2. The population size is set to 91 for MOEA/D and MOEA/D-NUMS and 92 for the others; whilst the maximum number of function evaluations is set to 5,520, i.e., approximately 60 generations. The DM is assumed to have three chances to elicit her/his preference information. Because the PF is unknown, only the R-HV metric is chosen in the performance assessment.

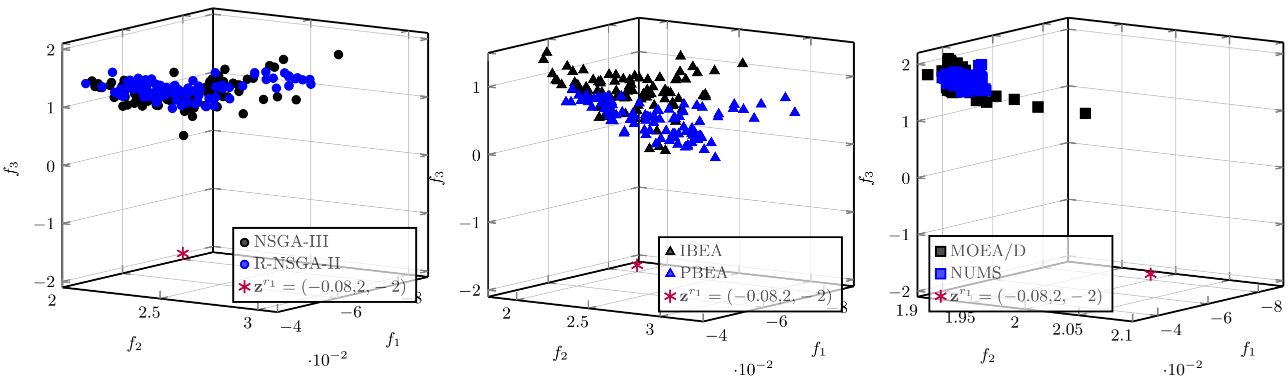

Figure 20: Solutions obtained on the 3-objective portfolio optimisation problem in the first interaction, where $\mathbf{z}^{r_1} = (-0.08, 2, -2)^T$.

Solutions obtained by different algorithms after three preference elicitations are presented in Figs. 20 to 22. More specifically, in the first preference elicitation, we assume that the DM is rather greedy. (S)he sets $\mathbf{z}^{r_1} = (-0.75, 3, -0.85)^T$ where each objective is as utopia as possible. As shown in Fig. 20, solutions obtained by the preference-based EMO algorithms are similar to their non-preference-based counterparts. It is interesting to note that the performance of MOEA/D is better

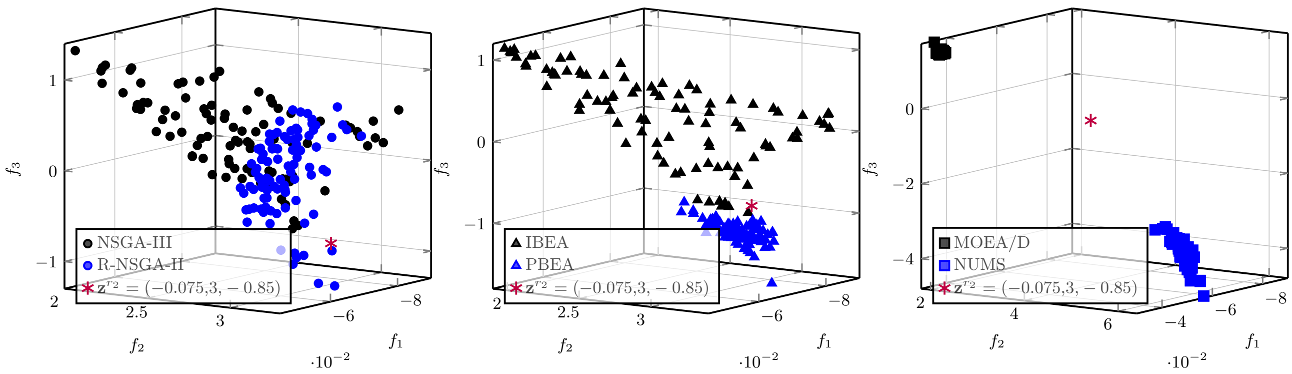

Figure 21: Solutions obtained on the 3-objective portfolio optimisation problem in the second interaction, where $\mathbf{z}^{r_2} = (-0.75, 3, -0.85)^T$.

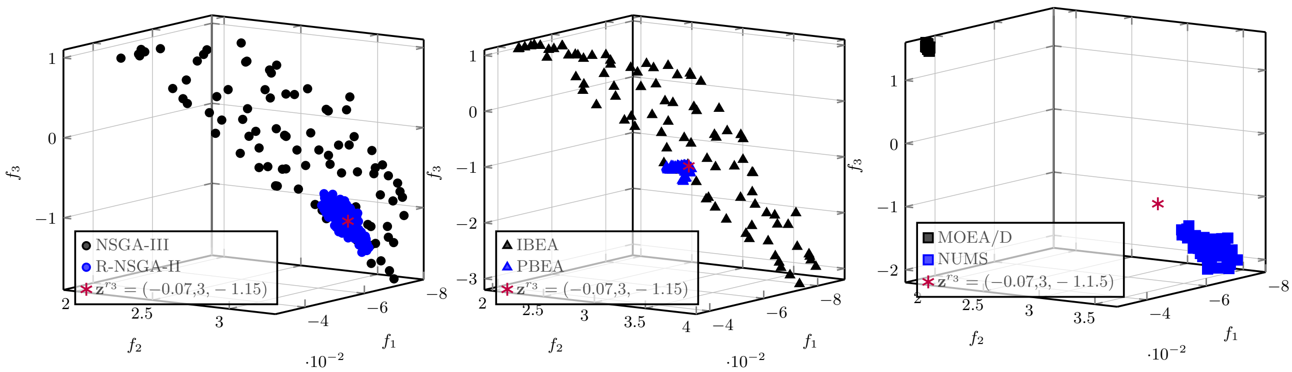

Figure 22: Solutions obtained on the 3-objective portfolio optimisation problem in the third interaction, where $\mathbf{z}^{r_3} = (-0.07, 3, -1.15)^T$.

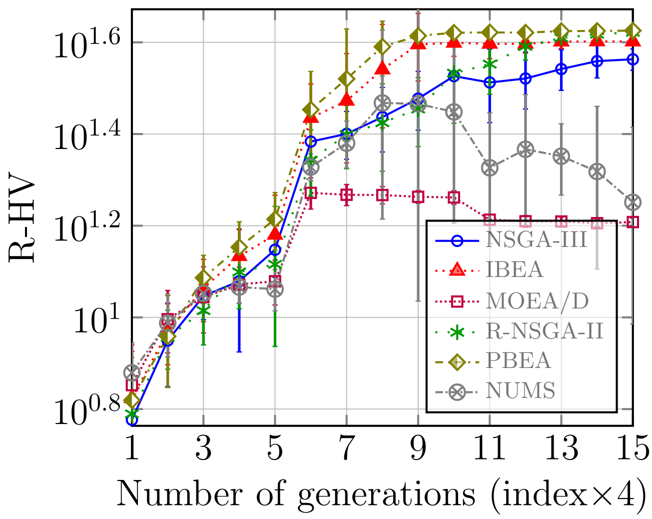

Figure 23: Trajectories of R-HV values versus the number of generations on the 3-objective portfolio optimisation problem.

than MOEA/D-NUMS according to the R-HV trajectories shown in Fig. 23. Furthermore, it is clear that almost all solutions are dominated by $\mathbf{z}^{r_1}$. This suggests that $\mathbf{z}^{r_1}$ is too utopia to achieve.

In the second preference elicitation, the DM made some modifications on some objectives and set $\mathbf{z}^{r_2} = (-0.75, 3, -0.85)^T$. As shown in Fig. 21, solutions found by R-NSGA-II and PBEA have a much better approximation to $\mathbf{z}^{r_2}$ this time whilst solutions obtained NSGA-III and IBEA do not change significantly. In addition, as shown in Fig. 23, the trajectories of R-HV values have a significant surge after the elicitation of $\mathbf{z}^{r_2}$. This is partially caused by using a more reasonable reference point to guide the preference-based EMO algorithms and also in the performance evaluation.

Moreover, we also notice that $\mathbf{z}^{r_2}$ is dominated by some solutions obtained by R-NSGA-II and PBEA, this suggests that some objectives deserve better expectation. Bearing this consideration in mind, the DM fine-tunes the reference point and set $\mathbf{z}^{r_3} = (-0.07, 3, -1.15)^T$ in the last preference elicitation. As shown in Fig. 22, solutions obtained by R-NSGA-II and PBEA have a decent approximation around $\mathbf{z}^{r_3}$. This is also reflected by their best R-HV values. In contrast, solutions found by NSGA-III and IBEA do not show significant difference with respect to the second preference elicitation. This suggests that they almost converge. Moreover, since $\mathbf{z}^{r_3}$ is almost on the PF manifold and the solutions obtained by NSGA-III and IBEA well approximate the whole PF, their R-HV values are also competitive. However, we notice that the performance of MOEA/D and MOEA/D-NUMS are even worse. This might be caused by the largely disparate scales of different objectives which make the simplex assumption of decomposition-based EMO algorithm fail to meet the actual shape of the PF.

In addition to the three objectives considered in the MVS model, investors may also consider the robustness and the portfolio return as additional objectives in their portfolio investments. As for the prior objective, we apply the kurtosis model proposed by Lai et al. [84] to evaluate the probability of extreme events. In particular, the larger the kurtosis is, the higher probability the extreme events occur. In other words, the corresponding portfolio investment is less robust. Specifically, the kurtosis can be calculated as:

$$\mathbb{K}[\psi(\mathbf{P})] = \mathbb{E}[(\psi(\mathbf{P}) - \mathbb{E}(\mathbf{P}))^4], \tag{13}$$

As for the portfolio return, it can be evaluated as the turnover of stock investments. In particular, a high turnover ratio indicates an active state of the underlying stock investments. Specifically, the turnover of a portfolio of financial assets $\mathbf{P}$ is calculated as:

$$\phi(\mathbf{P}) = \sum_{i=1}^{n} \rho_i t_i, \tag{14}$$

where t_i represents the turnover of each financial asset ρ_i, $i \in \{1, \cdots, n\}$. The expected turnover is calculated as:

$$\mathbb{E}(\phi(\mathbf{P})) = \sum_{i=1}^{n} \rho_i \mathbb{E}[t_i], \tag{15}$$

In summary, the Mean-Variance-Skewness-Kurtosis-Turnover (MVSKT) model, which constitutes a five-objective portfolio optimisation problem, is formulated as:

$$\begin{cases} \text{maximise} & \mathbb{E}[\psi(\mathbf{P})] \\ \text{minimise} & \mathbb{V}[\psi(\mathbf{P})] \\ \text{maximise} & \mathbb{S}[\psi(\mathbf{P})] \\ \text{minimise} & \mathbb{K}[\psi(\mathbf{P})] \\ \text{maximise} & \mathbb{E}[\phi(\mathbf{P})] \end{cases}, \tag{16}$$

In the experiments, almost all settings are the same as the 3-objective case except the population size and the number of function evaluations. In particular, the population size is set to 210 for MOEA/D and MOEA/D-NUMS and 212 for the others; whilst the maximum number of function evaluations is set to 12,720, i.e. approximately 60 generations in total. Figs. 24 to 26 plot the population distributions of solutions obtained by different algorithms after three preference elicitations.

Similar to the three-objective case, we assumed that the DM specifies a reference point which has an as utopia value as possible at each objective. From Fig. 24 and Fig. 27, we find that three preference-based EMO algorithms have shown similar performance in terms of population distribution and R-HV values. In particular, MOEA/D is the best algorithm under such preference setting.

In the second preference elicitation, the DM modified the aspiration at each objective, especially on the skewness and kurtosis. As shown in Fig. 27, all R-HV trajectories have experienced a significant surge after the second preference elicitation. This is similar to the observation in Fig. 23. It is also interesting to note that although the R-HV values of MOEA/D and MOEA/D-NUMS have been improved, their obtained solutions are not as satisfactory as the other peers. Especially for MOEA/D-NUMS, its obtained solutions do not have significant difference comparing to the first preference elicitation.

As shown in Fig. 27, the R-HV values were improved in the last preference elicitation. As discussed before, this is partially because the DM supplied reference point becomes more reasonable. As shown in Fig. 26, solutions found by R-NSGA-II and PBEA are close to $\mathbf{z}^{r_3}$. In contrast, solutions found by MOEA/D and MOEA/D-NUMS do not show significant difference with respect to the change of reference point.

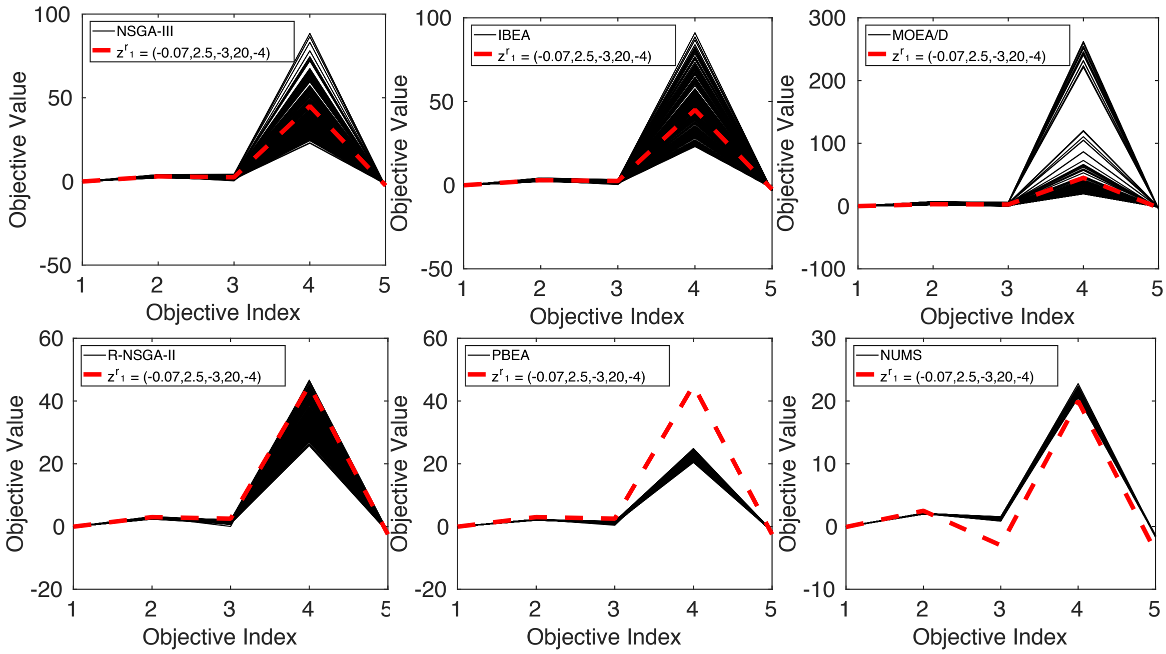

Figure 24: Solutions obtained on 5-objective portfolio optimisation problem in the first interaction, where $\mathbf{z}^{r_1} = (-0.07, 2.5, -3, 20, -4)^T$.

Answers to *RQ*4: From our experiments, we find that a preference-based EMO algorithm is able to respond to the change of the DM's preference information in an interactive manner. As discussed in Section 4.3, eliciting appropriate preference information is far from trivial, especially under a black-box setting. In other words, the DM may easily elicit an unrealistically utopian aspiration for each objective function at the outset. An interactive elicitation manner provides the DM with an avenue to progressively understand the underlying problem and adjust her/his preference information within limited computational budgets.

4.5 Using Preference-based EMO Algorithms as a General-Purpose Optimiser

In our previous experiments, the preference-based EMO algorithms are studied in a conventional way, i.e., used to approximate a ROI, which is normally a partial region of the PF. On the other hand, we come up with another question: *can we expect a preference-based EMO algorithm to be capable of approximating the whole PF if we set more than one ROI evenly spreading over the PF?* To address this question, we choose R-NSGA-II and PBEA as the representative preference-based EMO algorithms from dominance- and indicator-based frameworks in our experiments to compare with three non-preference-based EMO algorithms, i.e., NSGA-III, IBEA and MOEA/D. In particular, we do not

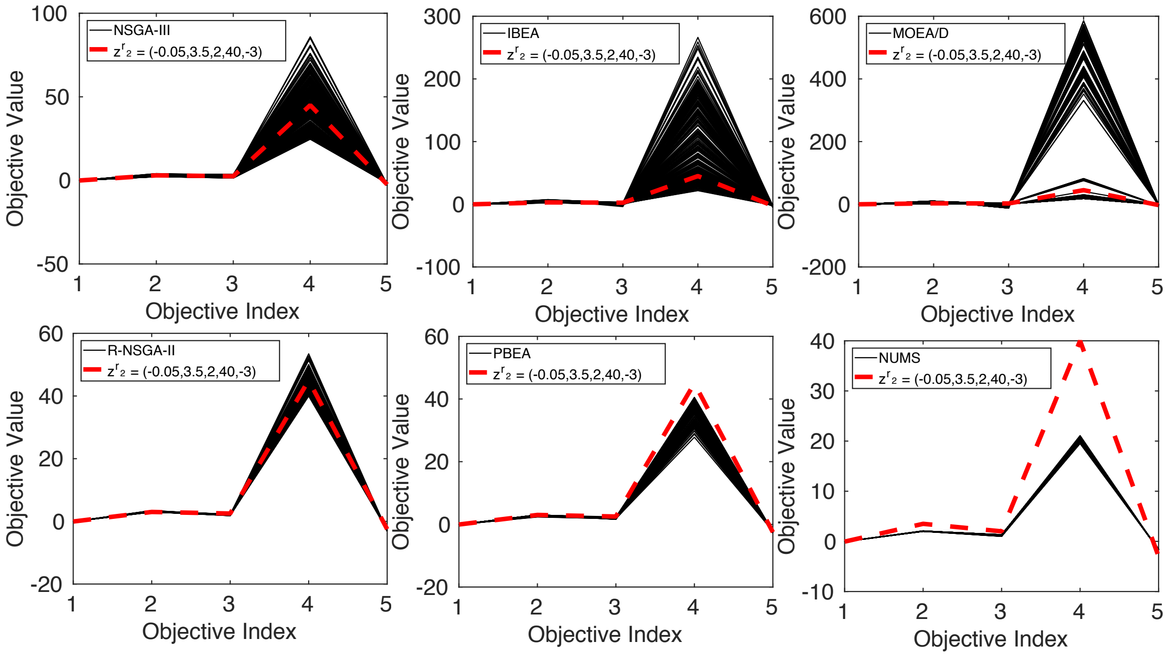

Figure 25: Solutions obtained on 5-objective portfolio optimisation problem in the second interaction, where $\mathbf{z}^{r_2} = (-0.05, 3.5, 2, 40, -3)^T$.

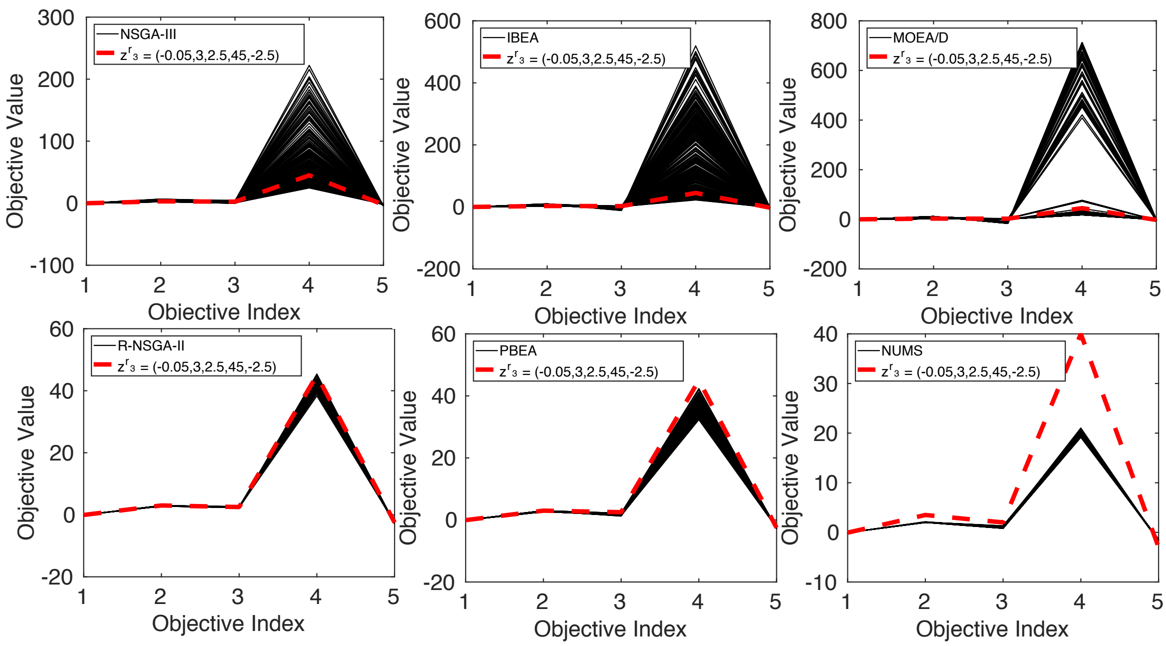

Figure 26: Solutions obtained on 5-objective portfolio optimisation problem in the third interaction, where $\mathbf{z}^{r_3} = (-0.05, 3, 2.5, 45, -2.5)^T$.

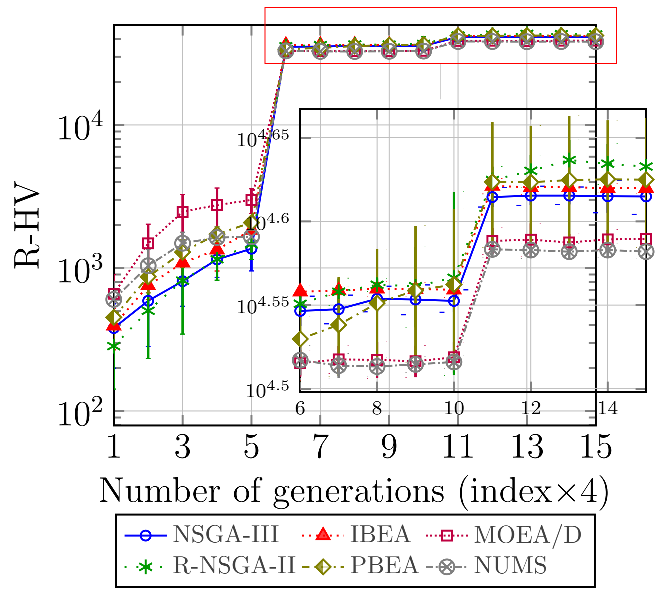

Figure 27: Trajectories of R-HV values versus the number of generations on the 5-objective portfolio optimisation problem.

consider MOEA/D-NUMS because it becomes an ordinary MOEA/D when used to approximate the whole PF; whilst g-NSGA-II and RMEAD2 are not considered given their poor performance reported in Sections 4.1 and 4.2. We notice that Liu et al. [85] proposed an many-objective EA by using a set of evolving reference points. Whereas reference points are used as the DM supplied *a priori* preference information over the whole PF to guide the search process in our experiments.

DTLZ and DTLZ^{-1} problems are used as the benchmark problems where the number of objectives is set as $m \in \{3, 5, 8, 10, 15, 25, 50, 100\}$. In our experiment, we use a set of evenly distributed weight vectors, as used in the decomposition-based EMO methods, to represent the preference that cover the whole PF. As discussed in our recent study on massive objective optimisation [86], it is very challenging to set evenly distributed weight vectors when $m \geq 25$; whilst we use the weight vector generation method proposed therein [86] to serve our purpose. Note that R-NSGA-II and PBEA are all able to handle more than one DM supplied reference point. Since we need to consider test problems with a massive number of objectives, the calculation of HV will be extremely time consuming even when using a Monte Carlo approximation [87]. In this case, only the IGD metric is considered in our experiment to evaluate the performance. In particular, the settings of IGD calculation for problems with $m < 25$ can be found in [88]; otherwise we use the settings suggested in [86]. Furthermore, the settings of the population size and the number of generations can be found in Section 1 the supplementary document.

In Figs. 28 and 29, we use heat maps of ranks of IGD metric values obtained by different algorithms to present the comparison results. From Fig. 28, we can see that NSGA-III, MOEA/D and R-NSGA-II are the most competitive algorithms for DTLZ1 to DTLZ4 problems; whilst IBEA is the worst one in most cases. Although the performance of PBEA is not promising when the number of objectives is relatively small, it gradually becomes more competitive with the increase of the number of objectives. As for the results on DTLZ1^{-1} to DTLZ4^{-1} shown in Fig. 29, R-NSGA-II becomes the most competitive algorithm whilst MOEA/D is almost the worst algorithm in most cases. This is not surprising given the poor performance of MOEA/D on DTLZ^{-1} problems reported in [81]. Moreover, we notice that the performance of NSGA-III, another decomposition-based algorithm, is not satisfactory as well. The performance of IBEA is slightly better than that of NSGA-III and MOEA/D, especially when the number of objectives is small. In contrast, PBEA, the other preference-based algorithm, has shown strong performance in most cases, especially when $m \geq 15$.

In principle, R-NSGA-II can be regarded as a decomposition-based algorithm when the DM supplied reference points are replaced by the weight vectors used in NSGA-III and MOEA/D. Their major difference lies in the way of how to evaluate the closeness of a solution to a weight vector. Specifically,

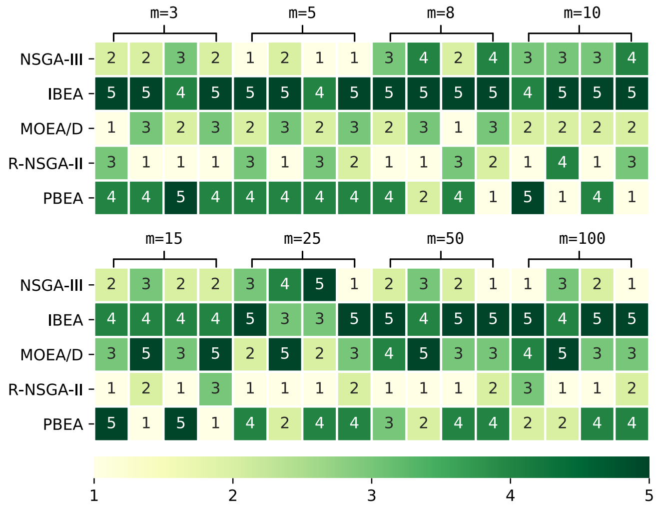

Figure 28: Heat maps of the ranks of IGD values obtained by different algorithms on DTLZ1 to DTLZ4 problems with various numbers of objectives.

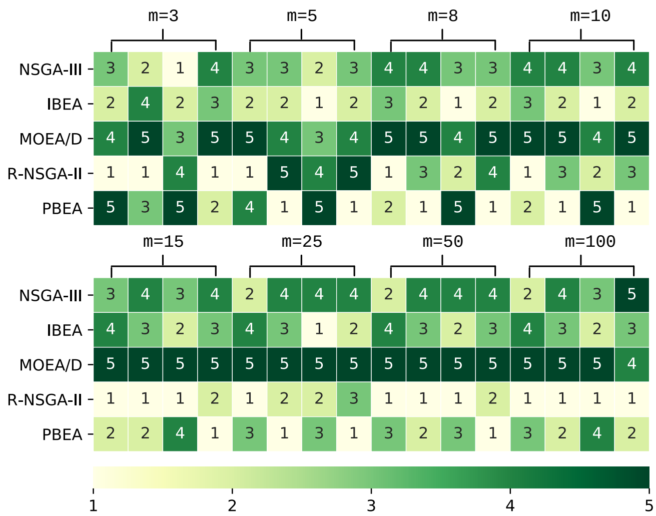

Figure 29: Heat maps of the ranks of IGD values obtained by different algorithms on DTLZ1^{-1} to DTLZ4^{-1} problems with various numbers of objectives.

it is evaluated as the perpendicular distance towards the reference line formed by the origin and a weight vector in NSGA-III. MOEA/D uses the Tchebycheff distance to evaluate the fitness of a solution. In R-NSGA-II, it uses the Euclidean distance between a solution and a weight vector as a major criterion in the environmental selection. It is interesting to note that R-NSGA-II has achieved the best IGD values in many cases whereas the distribution of its obtained solutions is not satisfactory such as two examples shown in Figs. 30 and 33. Even worse population diversity have been witnessed for solutions obtained by PBEA. These observations are not surprising as R-NSGA-II and PBEA do not have a well crafted diversity preservation mechanism and the using of more than one reference point may bring more uncertainty in selection process. We infer that its promising IGD values come from the better convergence of solutions towards the PF.

Answers to *RQ*5: From our experiments, we find that a preference-based EMO algorithm is able to approximate the whole PF, given that the DM supplied preference information is used in an appropriate manner. In particular, the direct Euclidean distance between the solution and the reference point, as in R-NSGA-II, is a surprisingly reliable metric to guide the optimisation process. By this means, preference elicitation become another decomposition method in EMO. In addition, R-NSGA-II is flexible to the orientation of the PF. Since there are more than one reference point,

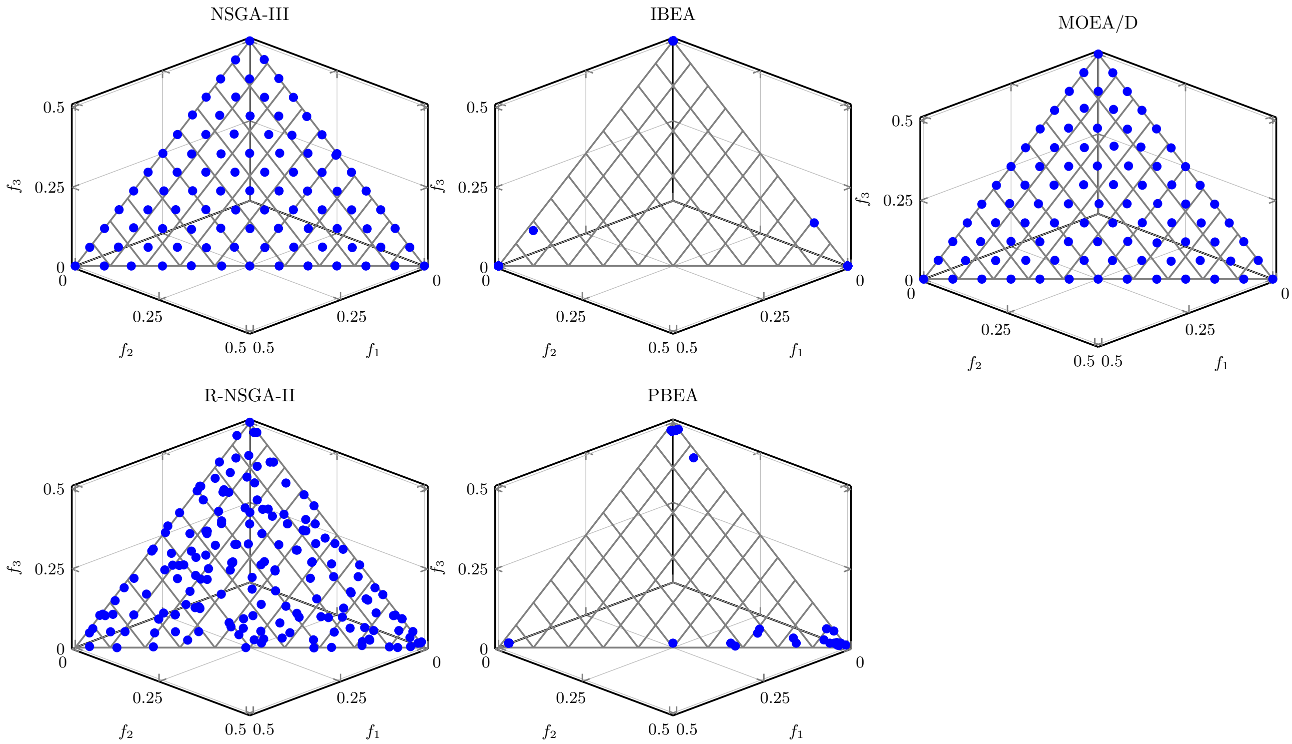

Figure 30: Solutions obtained by different algorithms on the DTLZ1 test problem with 3 objectives.

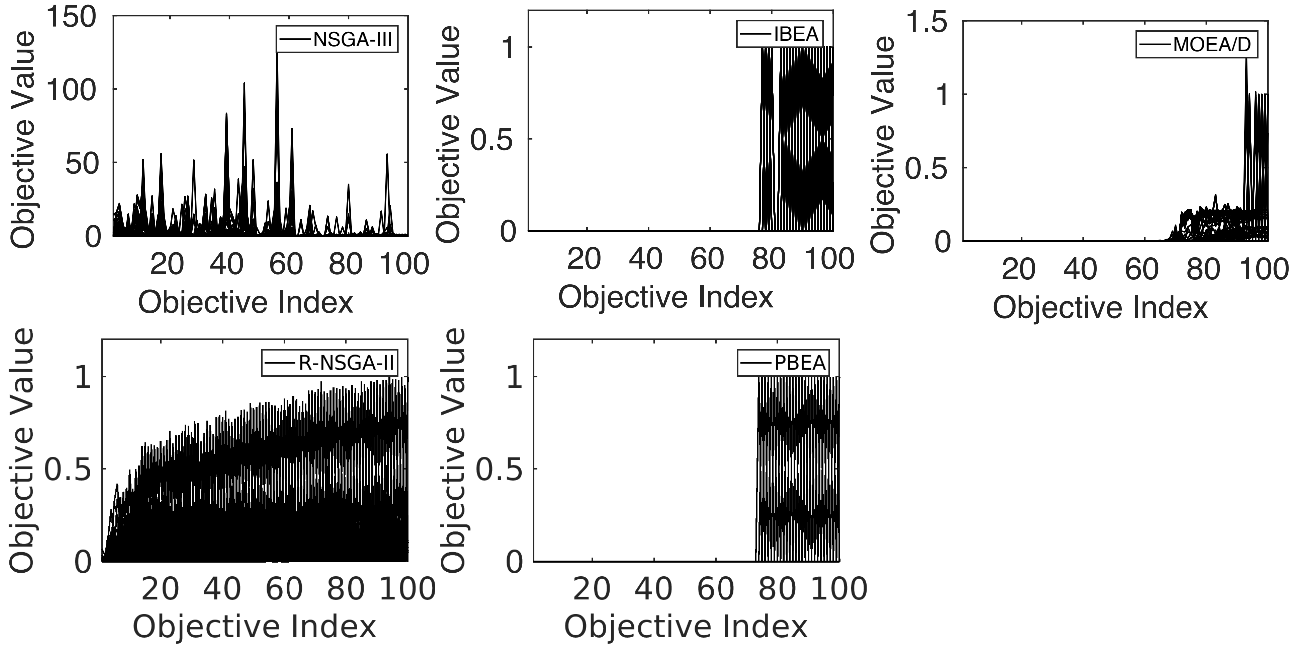

Figure 31: Solutions obtained by different algorithms on the DTLZ3 test problem with 100 objectives.

one key challenge is how to balance the search power across different reference points. Furthermore, it is also challenging to maintain the local diversity within a ROI specified by each reference point.

5 Conclusions and Future Works

Finding trade-off solution(s) most satisfying the DM's preference information is the ultimate goal of multi-objective optimisation in practice. This paper first provides a pragmatic review on the current developments of preference-based EMO. In particular, the literature review was mainly conducted according to the elicitation manner, i.e., when to ask the DM to elicit her/his preference informa-

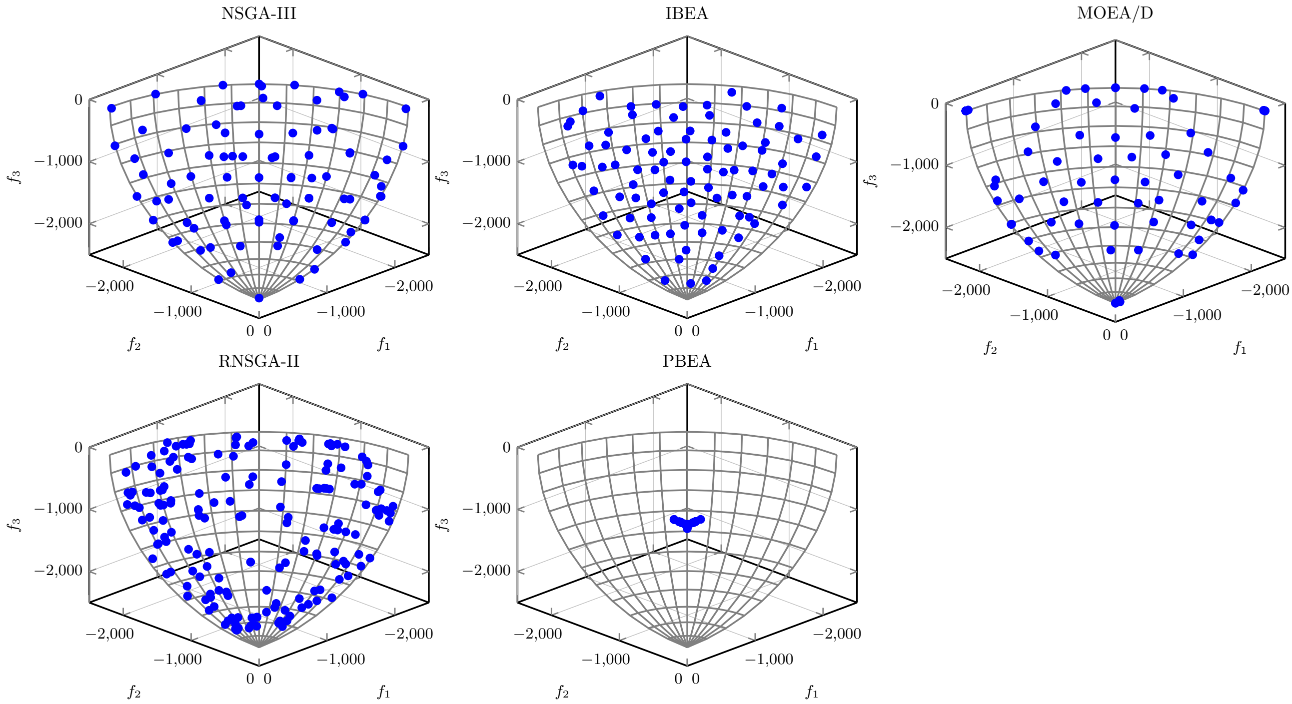

Figure 32: Solutions obtained by different algorithms on the minus DTLZ3 test problem with 3 objectives.

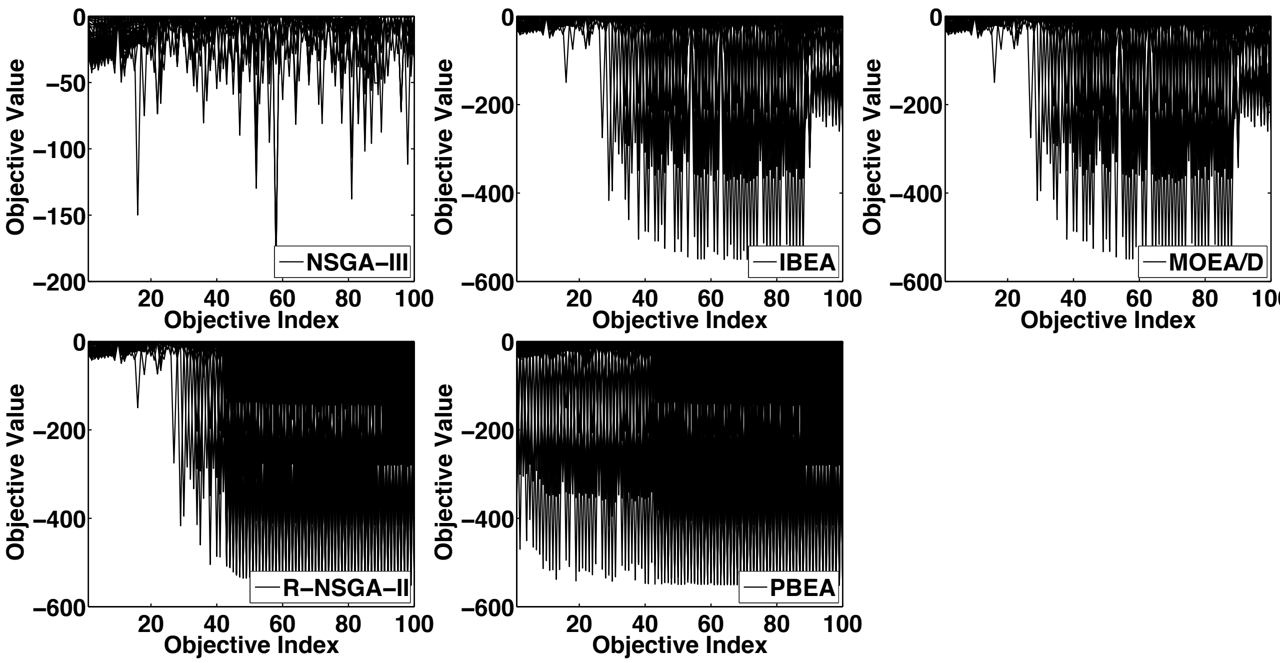

Figure 33: Solutions obtained by different algorithms on the minus DTLZ1 test problem with 100 objectives.

tion. Afterwards, we conduct a series of experiments to have a holistic comparison of six prevalent preference-based EMO algorithms against three iconic EMO algorithms without considering any preference information under various settings. In summary, we come up with the following five major observations.

- *A well designed EMO algorithm, without considering any DM's preference information, is able to be competitive for finding the SOI.* This is particularly true when the number of objectives is small or problems are with correlated objectives. However, this becomes significantly more difficult, if not impossible, with the increase of the number of objectives.

- *Dominance-, indicator- and decomposition-based frameworks all can be used as a baseline for designing effective preference-based EMO algorithms.* Distance metric, such as Euclidean distance and Tchebycheff distance, is a reliable way to transform the DM supplied preference information into the selection pressure in an algorithm. On the other hand, if the DM's preference information is not well utilised, it brings more uncertainty to the search process thus makes the end algorithm even worse than those non-preference-based counterparts.

- *A preference-based EMO algorithm may fail to find the ROI if the DM elicits unreasonable preference information.* Note that this is not uncommon when encountering a real-world black-box system of which the DM has little knowledge.

- *Interactive preference elicitation provide a better opportunity for the DM to progressively understand the underlying black-box system thus to gradually rectify her/his preference information.*

- *Preference elicitation can be used as another means of decomposition method in EMO.* That is to say, a preference-based EMO algorithm, e.g., R-NSGA-II, is able to approximate the whole PF given that the DM supplied preference information aim to cover the whole PF instead of a partial region.

EMO and MCDM are actually sibling communities which share many overlaps. However, they have been extensively developed in parallel in the past three decades. Although there is a growing trend of seeking the synergy between them, as introduced in Section 1, it is still a lukewarm whilst more efforts are required along this line of research. This paper lights up the potential convergence between EMO and MCDM under the same paradigm. Many questions are still open for future exploration whilst we just name a few as follows:

- This paper only investigate the case where the DM's preference information is represented as a reference point. As discussed in Section 2, there are various other ways to represent and model the DM's preference information, such as value function [13], holistic comparisons [60], fuzzy linguistic terms [45]. It is interesting to investigate a universal framework that is able to embrace different preference information.

- Furthermore, it is far from trivial to quantitatively compare the performance of different preference-based EMO algorithms when they use different ways to represent preference information. In particular, the R-metric used in this paper can only be useful when a reference point is used to represent the DM's preference information. It is interesting to develop other performance metric for a wider range of preference representations.

- As discussed in Section 4.4, interactive EMO is a promising way to progressively approximate the SOI with the assistance of DM(s) under a black-box setting. It is worthwhile to invest more efforts to develop a human-in-the-loop optimisation paradigm in future. For example, it is interesting and important to study effective human-computer interaction mechanism to engage with DMs for better preference elicitation. Furthermore, due to the lack of data collected during the interaction process, it is challenging to train a reliable preference model. Research on few-shot learning [89] will be valuble to facilitate the preference modeling building from *small* data. Last but not the least, it is not uncommon to have inconsistency during the deicion-making process [62]. Therefore, it is interesting to take this uncertainty information into account in both preference modelling and optimisation.

Data availability: The the supplementary document and source codes can be found from our project page: `https://github.com/COLA-Laboratory/empirical_preference`.

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